# Econometrics

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# Introduction

# Who Uses Econometrics

Statistical analysis of economic data:

- Economics
- Finance
- Business
- Consulting
- Government

# What Makes Econometrics Special

Econometrics is not just "statistics using economic data."

Special issues related to the properties of economic data.

- No experiments; only "observational data"
- Special issues and features that arise routinely in economic data
- Predictive modeling, causal modeling

# Types of Recorded Economic Data

- Continuous vs. discrete
- Time series vs. cross section
- Panel

Complement: Explore nominal, ordinal, interval and ratio data

# Web Data Resources

- Resources for Economists (AEA)
- FRED (Federal Reserve Economic Data)
- National Bureau of Economic Research
- Quandl
- FRB Phila Real-Time Data Research Center

# Software

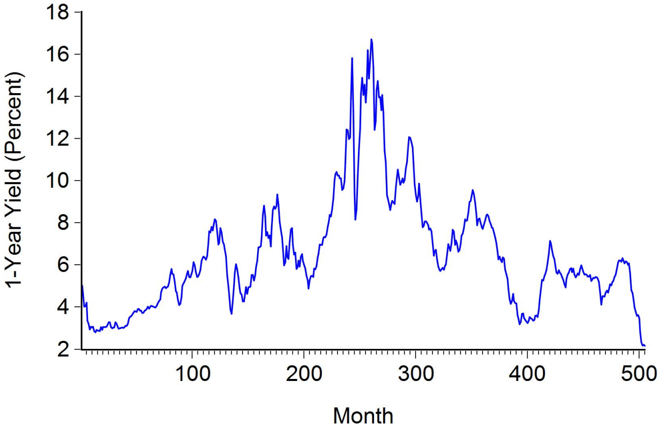
- R
  - CRAN
  - RStudio
  - R-bloggers
- More: Eviews, Python
- Still more: *Econometrics Journal* software links

# Graphics

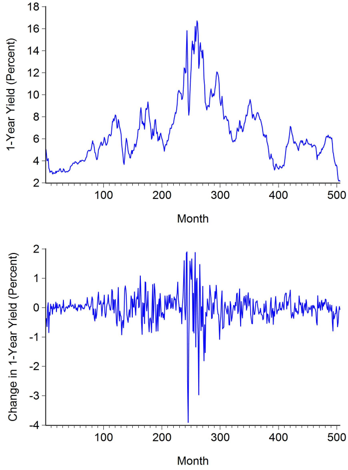
# Graphics

Let's have some fun and look at the pictures *first*...

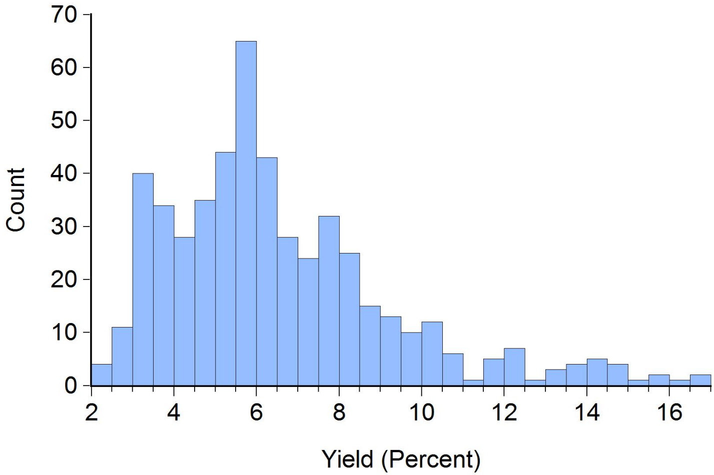
# Time Series Plot:
# 1-Year Goverment Bond Yield, Levels



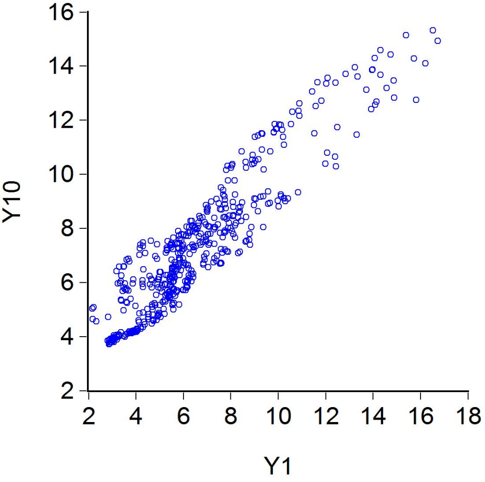

# Time Series Plot:
# 1-Year Goverment Bond Yield, Levels and Changes



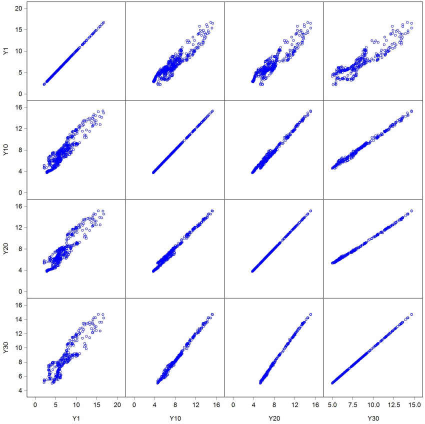

# Histogram: 1-Year Government Bond Yield




# Bivariate Scatterplot
# 1-Year and 10-Year Government Bond Yields



# Scatterplot Matrix:
# 1-, 10-, 20- and 30-Year Government Bond Yields




# Graphics

- Summarize and reveal patterns in univariate time-series data. Time Series plots. Trend, seasonal, cycle, outliers, ...
- Summarize and reveal patterns in univariate cross-section data. Histograms are helpful for learning about distributional shape. Symmetric, skewed, fat-tailed, ...
- Identify relationships and understand their nature, in both multivariate time-series and multivariate cross-section environments. Bivariate scatterplots. Does a relationship exist? Is it linear or nonlinear? Are there outliers?
- Identify relationships and understand their nature in panel data. Cross-sectional histograms across time periods, or time series plots across cross-sectional units.
- Compare different pieces of data via multiple comparisons. Scatterplot matrix.

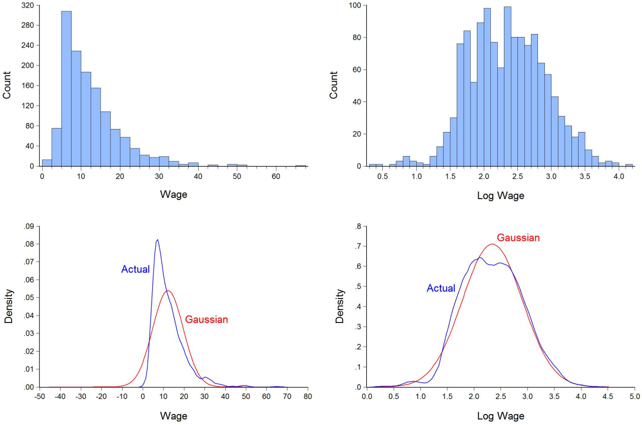
# Univariate and Multivariate Graphics

- Time-series plot
  - levels
  - change
- Density estimate
  - histogram
  - smoothed
- Scatterplot
  - Two-way
  - Multi-way

# Principles of Graphical Style

- Know your audience, and know your goals.
- Appeal to the viewer.
- Show the data, and only the data, withing the bounds of reason.
  - Avoid distortion. The sizes of effects in graphics should match their size in the data. Use common scales in multiple comparisons.
  - Minimize, within reason, non-data ink. Avoid chartjunk.
  - Third, choose aspect ratios to maximize pattern revelation. Bank to 45 degrees.
  - Maximize graphical data density.
- Revise and edit, again and again (and again). Graphics produced using software defaults are almost *never* satisfactory.

# Distributions of Wages and Log Wages



# Probability and Statistics Review

# "Sample" EPC: Simple vs. Partial Correlation

(Read them all carefully!)

The set of pairwise scatterplots that comprises a multiway scatterplot provides useful information about the joint distribution of the set of variables, but it's incomplete information and should be interpreted with care. A pairwise scatterplot summarizes information regarding the **simple correlation** between, say, $x$ and $y$. But $x$ and $y$ may appear highly related in a pairwise scatterplot even if they are in fact unrelated, if each depends on a third variable, say $z$. The crux of the problem is that there's no way in a pairwise scatterplot to examine the correlation between $x$ and $y$ *controlling* for $z$, which we call **partial correlation**. When interpreting a scatterplot matrix, keep in mind that the pairwise scatterplots provide information only on simple correlation.

# Moments, Sample Moments and Their Sampling Distributions

- Discrete random variable, $y$
- Discrete probability distribution $p(y)$
- Continuous random variable $y$
- Probability density function $f(y)$

# Population Moments: Expectations of Powers of R.V.'s

Mean measures location:

$$\mu = E(y) = \sum_i p_i y_i \ \ \text{(discrete case)}$$

$$\mu = E(y) = \int y\, f(y)\, dy \ \ \text{(continuous case)}$$

Variance, or standard deviation, measures dispersion, or scale:

$$\sigma^2 = var(y) = E(y - \mu)^2.$$

– $\sigma$ easier to interpret than $\sigma^2$. Why?

# More Population Moments

Skewness measures skewness (!)

$$S = \frac{E(y-\mu)^3}{\sigma^3}.$$

Kurtosis measures tail fatness relative to a Gaussian distribution.

$$K = \frac{E(y-\mu)^4}{\sigma^4}.$$

# Covariance and Correlation

Multivariate case: Joint, marginal and conditional distributions
$f(x,y)$, $f(x)$, $f(y)$, $f(x|y)$, $f(y|x)$

Covariance measures linear dependence:

$$cov(y,x) = E[(y_t - \mu_y)(x_t - \mu_x)].$$

So does correlation:

$$corr(y,x) = \frac{cov(y,x)}{\sigma_y \sigma_x}.$$

Correlation is often more convenient. Why?

# Sampling and Estimation

$$\text{Sample}: \ \{y_i\}_{i=1}^{N} \sim f(y)$$

Sample mean:

$$\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i$$

Sample variance:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{N} (y_i - \bar{y})^2}{N}$$

Unbiased sample variance:

$$s^2 = \frac{\sum_{i=1}^{N} (y_i - \bar{y})^2}{N - 1}$$

# More Sample Moments

Sample skewness:

$$\hat{S} = \frac{\frac{1}{N}\sum_{i=1}^{N}(y_t - \bar{y})^3}{\hat{\sigma}^3}$$

Sample kurtosis:

$$\hat{K} = \frac{\frac{1}{N}\sum_{i=1}^{N}(y_t - \bar{y})^4}{\hat{\sigma}^4}$$

# Still More Sample Moments

Sample covariance:

$$\widehat{cov}(y,x) = \frac{1}{N}\sum_{i=1}^{N}[(y_i - \bar{y})(x_i - \bar{x})]$$

Sample correlation:

$$\widehat{corr}(y,x) = \frac{\widehat{cov}(y,x)}{\hat{\sigma}_y \hat{\sigma}_x}$$

# Exact Sampling Distribution of the Sample Mean (Requires *iid* Normality)

$$\text{Simple random sampling} : y_i \sim iid\, N(\mu, \sigma^2), i = 1, ..., N$$

$\bar{y}$ is unbiased, consistent, normally distributed with variance $\sigma^2/N$, and minimum variance unbiased (MVUE).

$$\bar{y} \sim N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$\sqrt{N}(\bar{y} - \mu) \sim N(0, \sigma^2)$$

$$\mu \in \left[\bar{y} \pm t_{1-\frac{\alpha}{2}}(N-1)\frac{s}{\sqrt{N}}\right] \; w.p. \; \alpha$$

$$\mu = \mu_0 \implies \frac{\bar{y} - \mu_0}{\frac{s}{\sqrt{N}}} \sim t(N-1)$$

# Approximate Asymptotic Sampling Distribution (Does Not Require Normality)

$$\text{Simple random sampling : } y_i \sim iid(\mu, \sigma^2), i = 1, ..., N$$

$\bar{y}$ is unbiased, consistent, asymptotically normally distributed with variance $\sigma^2/N$, and best linear unbiased (BLUE).

$$\bar{y} \overset{a}{\sim} N\left(\mu, \frac{\sigma^2}{N}\right)$$

$$\sqrt{N}(\bar{y} - \mu) \to_d N(0, \sigma^2)$$

$$\text{As } N \to \infty, \ \mu \in \left[\bar{y} \pm z_{1-\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{N}}\right] \ w.p. \ \alpha$$

$$\text{As } N \to \infty, \ \frac{\bar{y} - \mu_0}{\frac{\hat{\sigma}}{\sqrt{N}}} \sim N(0, 1)$$

# Notational Aside

Standard cross-section notation: $i = 1, ..., N$

Standard time-series notation: $t = 1, ..., T$

Much of our discussion will be valid in *both* cross-section and time-series environments, but still we have to pick a notation.

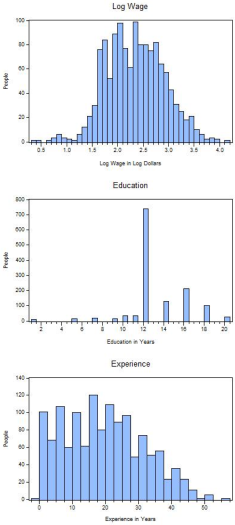
Without loss of generality, we will use $t = 1, ..., T$.

# Regression

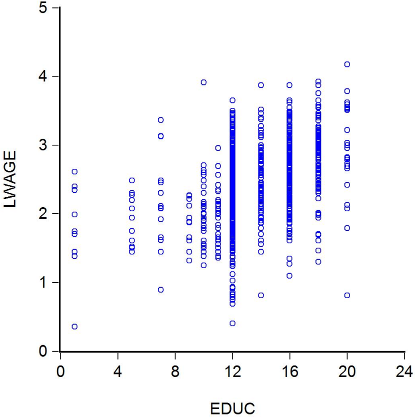
# Regression

Surely the all-time greatest statistical and econometric workhorse...

# Distributions of Log Wage, Education and Experience




# Scatterplot: Log Wage vs. Education



# Regression as Curve Fitting

Fit a line:

$$y_t = \beta_1 + \beta_2 x_t$$

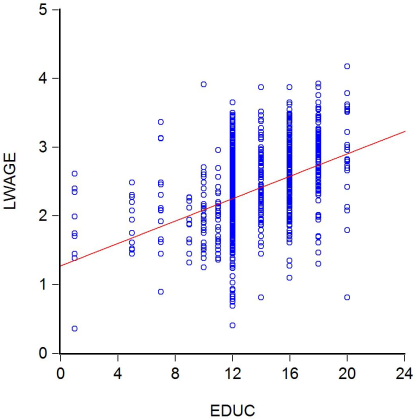
Solve:

$$\min_\beta \sum_{t=1}^{T} (y_t - \beta_1 - \beta_2 x_t)^2$$

$\beta$ is the set of two parameters $\beta_1$ and $\beta_2$

$\hat{\beta}$ is the set of fitted parameters $\hat{\beta}_1$ and $\hat{\beta}_2$

# Scatterplot: Log Wage vs. Education with Superimposed Regression Line




$$\widehat{LWAGE} = 1.273 + .081EDUC$$

# Actual Values, Fitted Values and Residuals

The fitted values are

$$\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 x_t,$$

$$t = 1, ..., T.$$

The residuals are the difference between actual and fitted values,

$$e_t = y_t - \hat{y}_t,$$

$$t = 1, ..., T.$$

# Multiple Linear Regression ($K$ RHS Variables)

Solve:

$$\min_{\beta} \sum_{t=1}^{T} (y_t - \beta_1 - \beta_2 x_{2t} - ... - \beta_K x_{Kt})^2$$

Fitted line:

$$\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 x_{2t} + \hat{\beta}_3 x_{3t} + ... + \hat{\beta}_K x_{Kt}$$

More compactly:

$$\hat{y}_t = \sum_{i=1}^{K} \hat{\beta}_i x_{it},$$

where $x_{1t} = 1$ for all $t$.

Wage dataset:

$$\widehat{LWAGE} = .867 + .093 EDUC + .013 EXPER$$

# Regression as a Probability Model

$$y_t = \beta_1 + \beta_2 x_{2t} + ... + \beta_K x_{Kt} + \varepsilon_t$$

$$\varepsilon_t \sim iidN(0, \sigma^2),$$

$$t = 1, ..., T.$$

Note:

$$E(y_t | x_t = x_t^*) = \beta_1 + \beta_2 x_{2t}^* + ... + \beta_K x_{Kt}^*$$

Estimation:

$$\min_\beta \sum_{t=1}^{T} \varepsilon_t^2$$

# Some Crucial Matrix Notation

You already understand matrix ("spreadsheet") notation
although you may not know it!

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} \quad X = \begin{pmatrix} 1 & x_{21} & x_{31} & \dots & x_{K1} \\ 1 & x_{22} & x_{32} & \dots & x_{K2} \\ \vdots & & & & \\ 1 & x_{2T} & x_{3T} & \dots & x_{KT} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix} \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{pmatrix}$$

# Elementary Matrices and Matrix Operations

$$\underline{0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Transposition: $A'_{ij} = A_{ji}$

Addition: For $A$ and $B$ $n \times m$, $(A + B)_{ij} = A_{ij} + B_{ij}$

Multiplication: For $A$ $n \times m$ and $B$ $m \times p$, $(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$.

Inversion: For non-singular $A$ $n \times n$, $A^{-1}$ satisfies $A^{-1}A = AA^{-1} = I$. Many algorithms exist for calculation.

# We Used to Write This:

$$y_t = \beta_1 + \beta_2 x_{2t} + ... + \beta_K x_{Kt} + \varepsilon_t$$

$$\varepsilon_t \sim iidN(0, \sigma^2)$$

$$t = 1, 2, ..., T$$

# Now, Equivalently, We Write This:

$$y = X\beta + \varepsilon \quad (1)$$

$$\varepsilon \sim N(\underline{0}, \sigma^2 I) \quad (2)$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} 1 & x_{21} & x_{31} & \dots & x_{K1} \\ 1 & x_{22} & x_{32} & \dots & x_{K2} \\ \vdots & & & & \\ 1 & x_{2T} & x_{3T} & \dots & x_{KT} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_K \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{pmatrix} \sim N\left( \begin{pmatrix} 0_1 \\ 0_2 \\ \vdots \\ 0_T \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma^2 \end{pmatrix} \right) \quad (2)$$

# The Full Ideal Conditions (FIC)

1. The true data-generating process is:

$$y = X\beta + \varepsilon$$

$$\varepsilon \sim N(\underline{0}, \sigma^2 I),$$

   and the fitted model matches it exactly.

   1.1 The relationship, if any, is truly linear, with no omitted variables, no measurement error, etc.
   1.2 The coefficients, $\beta$, are fixed.
   1.3 $\varepsilon \sim N$.
   1.4 The $\varepsilon_t$'s have constant variance $\sigma^2$.
   1.5 The $\varepsilon_t$'s are uncorrelated.

2. There is no redundancy among the variables contained in $X$, so that $X'X$ is non-singular.
3. $X$ is a non-stochastic matrix, fixed in repeated samples.

Surely these are heroic assumptions in economic environments. Much of econometrics (and this course) is devoted to relaxing them.

# Results

The OLS estimator is:

$$\hat{\beta}_{LS} = (X'X)^{-1}X'y$$

Under the full ideal conditions it is unbiased, consistent, normally distributed with covariance matrix $\sigma^2(X'X)^{-1}$, and MVUE.

We write:

$$\hat{\beta}_{LS} \sim N\left(\beta,\ \sigma^2(X'X)^{-1}\right),$$

or equivalently,

$$\sqrt{T}(\hat{\beta} - \beta) \sim N\left(0, \sigma^2\left(\frac{X'X}{T}\right)^{-1}\right).$$

# Regression Analysis of Wages, Education and Experience

Equation: UNTITLED Workfile: GRAPHS::Untitled\

View | Proc | Object | Print | Name | Freeze | Estimate | Forecast | Stats | Resids

Dependent Variable: LWAGE
Method: Least Squares
Date: 06/27/13 Time: 16:38
Sample (adjusted): 1 1323
Included observations: 1323 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.867382 | 0.075331 | 11.51431 | 0.0000 |
| EDUC | 0.093229 | 0.005045 | 18.48002 | 0.0000 |
| EXPER | 0.013104 | 0.001164 | 11.26208 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.232224 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.231061 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.492318 | Akaike info criterion | 1.422881 |
| Sum squared resid | 319.9376 | Schwarz criterion | 1.434644 |
| Log likelihood | -938.2358 | Hannan-Quinn criter. | 1.427291 |
| F-statistic | 199.6260 | Durbin-Watson stat | 1.926045 |
| Prob(F-statistic) | 0.000000 | | |

EXPER

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# "Top Matter": Background Information

- Dependent variable
- Method
- Date
- Sample
- Included observations

# “Middle Matter”: Estimated Regression Function

- Variable
- Coefficient
- Standard error
- *t*-statistic
- p-value

# "Bottom Matter: Statistics"

There are many...

# Regression Statistics: Mean dependent var 2.342

$$\bar{y} = \frac{1}{T} \sum_{t=1}^{T} y_t$$

# Regression Statistics: S.D. dependent var .561

$$SD = \sqrt{\frac{\sum_{t=1}^{T}(y_t - \bar{y})^2}{T-1}}$$

# Regression Statistics: Sum squared resid 319.938

$$SSR = \sum_{t=1}^{T} e_t^2$$

# Regression Statistics: Log likelihood -938.236

- Likelihood
- Log likelihood
- Maximum-likelihood estimation
- Hypothesis tests and model selection

# Regression Statistics: $F$-statistic 199.626

$$F = \frac{(SSR_{res} - SSR)/(K-1)}{SSR/(T-K)}$$

# Regression Statistics: S.E. of regression .492

$$s^2 = \frac{\sum_{t=1}^{T} e_t^2}{T - K}$$

$$SER = \sqrt{s^2} = \sqrt{\frac{\sum_{t=1}^{T} e_t^2}{T - K}}$$

# Regression Statistics: $R$-squared .232

$$R^2 = 1 - \frac{\sum_{t=1}^{T} e_t^2}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$

# Regression Statistics: Adjusted $R$-squared .231

$$\bar{R}^2 = 1 - \frac{\frac{1}{T-K}\sum_{t=1}^{T} e_t^2}{\frac{1}{T-1}\sum_{t=1}^{T}(y_t - \bar{y})^2}$$

# Regression Statistics: Schwarz criterion 1.435

$$SIC = T^{\left(\frac{K}{T}\right)} \frac{\sum_{t=1}^{T} e_t^2}{T}$$

# Regression Statistics: Akaike info criterion 1.423

$$AIC = e^{\left(\frac{2K}{T}\right)} \frac{\sum_{t=1}^{T} e_t^2}{T}$$

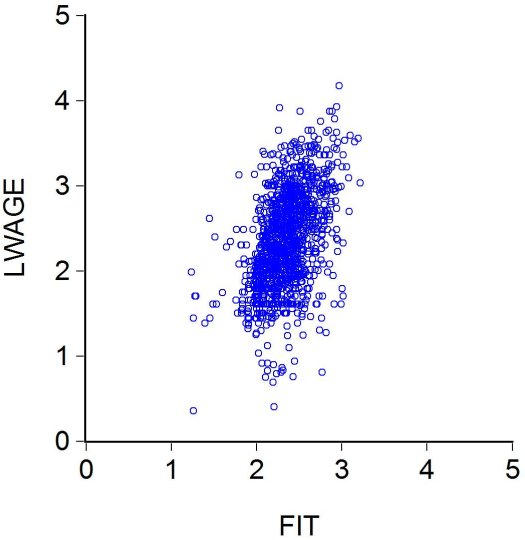
# Regression Statistics: Durbin-Watson stat. 1.926

$$\varepsilon_t = \phi \varepsilon_{t-1} + v_t$$

$$v_t \sim iidN(0, \sigma^2)$$

$$DW = \frac{\sum_{t=2}^{T} (e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2}$$

# Residual Scatter



# Residual Plot

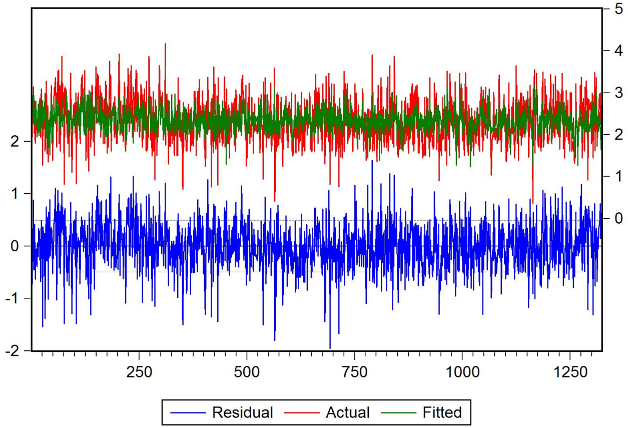
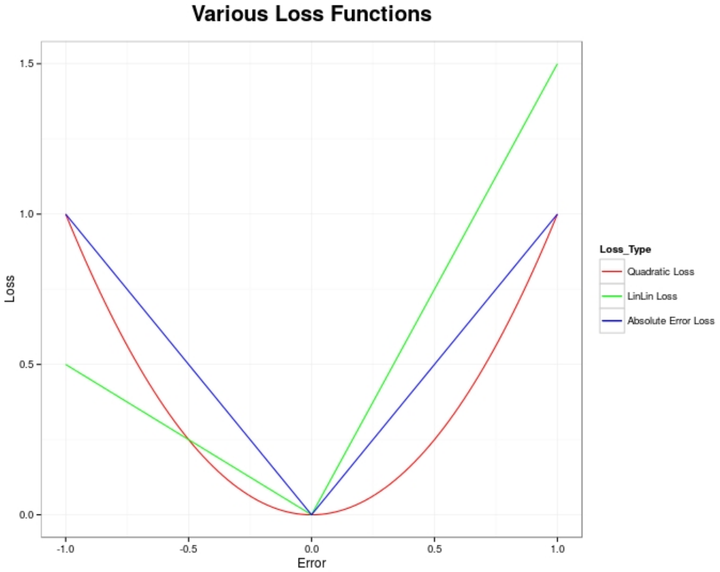



Figure: Wage Regression Residual Plot

# Beyond OLS: Non-Quadratic Objectives




# Ordinary Least Squares (OLS)

Recall that the OLS estimator, $\hat{\beta}_{OLS}$, solves:

$$\min_{\beta} \sum_{t=1}^{T} (y_t - \beta_1 - \beta_2 x_{2t} - \ldots - \beta_K x_{Kt})^2 = \min_{\beta} \sum_{t=1}^{T} \varepsilon_t^2$$

– Simple
(analytic closed-form expression, $(X'X)^{-1}X'y$)

– Wonderful properties under FIC
(Unbiased, consistent, Gaussian, MVUE)

But other approaches are possible and sometimes useful.

# Least Absolute Deviations (LAD)

The LAD estimator, $\hat{\beta}_{LAD}$, solves:

$$\min_{\beta} \sum_{t=1}^{T} |\varepsilon_t|$$

– Not as simple as OLS, but still simple
(Solves a linear programming problem)

– Useful properties under some violations of FIC
(Robust to outliers; more on that later)

– But there's a much bigger reason to be interested

# Conditional Mean and Median Functions

– OLS fits the conditional mean function:

$$mean(y|X) = x\beta$$

– LAD fits the conditional median function (50% quantile):

$$median(y|X) = x\beta$$

– The two are equal under symmetry as with FIC, but not under asymmetry, in which case the median is a better measure of central tendency

# Quantile Regression (QR)

Objective like LAD but unequal slopes on each side of 0.

QR estimator $\hat{\beta}_{QR}$ minimizes "linlin loss," or "check function loss":

$$\min_{\beta} \sum_{t=1}^{T} linlin(\varepsilon_t),$$

where:

$$linlin(e) = \begin{cases} a|e|, \text{ if } e \leq 0 \\ \\ b|e|, \text{ if } e > 0 \end{cases}$$

$$= a|e|\, I(e \leq 0) + b|e|\, I(e > 0).$$

$I(x) = 1$ if $x$ is true, and $I(x) = 0$ otherwise.

"$I(\cdot)$" stands for "indicator" variable.

"linlin" refers to linearity on each side of the origin.

Not as simple as OLS, but still simple
(solves a linear programming problem)

# What Does Quantile Regression Fit?

– QR fits the $d \cdot 100\%$ quantile:

$$quantile_d(y|X) = x\beta$$

where

$$d = \frac{b}{a+b} = \frac{1}{1+a/b}$$

– Median regression (LAD) is special case of $d = .5$

– Important generalization of median regression
(e.g., How do the wages of people in the far left tail of the wage distribution vary with education and experience, and how does that compare to those in the center of the wage distribution?)

# Indicator Variables in Cross Sections: Group Effects

# Dummy Variables for Group Effects

A dummy variable, or indicator variable, is just a 0-1 variable that indicates something, such as whether a person is female:

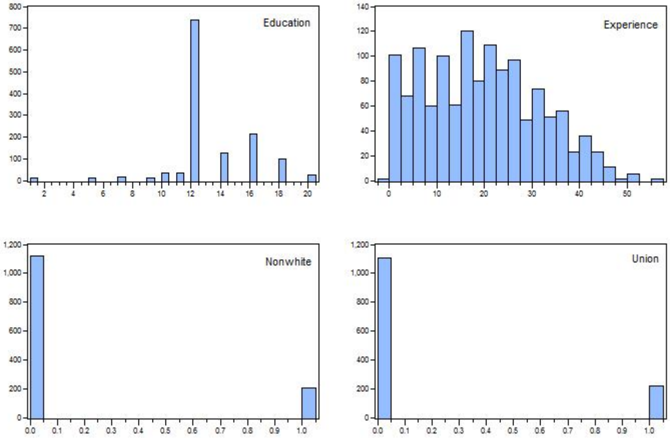
$$FEMALE_t = \begin{cases} 1 \text{ if person } t \text{ is female} \\ 0 \text{ otherwise} \end{cases}$$

(It really is that simple.)

"Intercept dummies"

Note that the sample mean of a dummy variable is the fraction of the sample with the indicated attribute.

# Histograms for Wage Covariates



# Important Issues

- The intercept corresponds to the "base case" across all dummies (i.e., when all dummies are simultaneously 0), and the dummy coefficients give the extra effects (i.e., when the respective dummies are 1).

- Alternatively, use a full set of dummies for each category (e.g., both a union dummy and a non-union dummy) and drop the intercept. (More useful/common for in time-series situations)

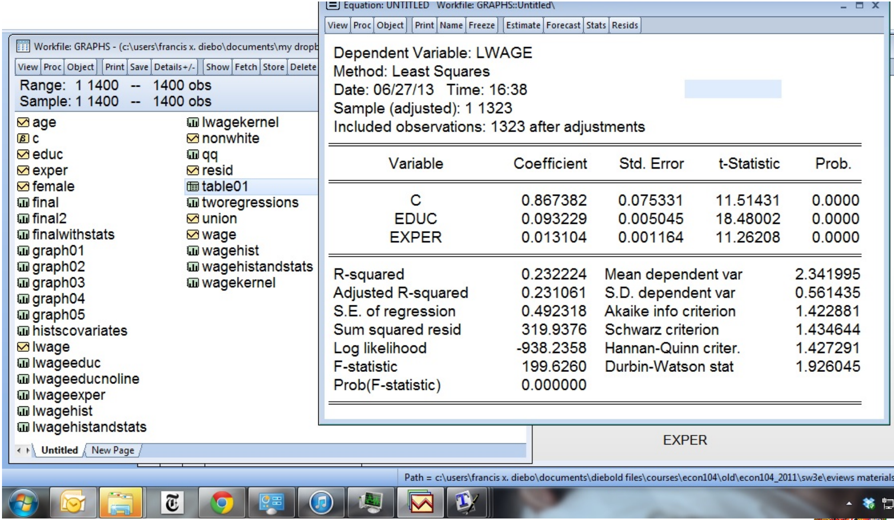
- Never include a full set of dummies *and* an intercept. Would be totally redundant: "Perfect Multicollinearity"

# Controlling for Sex, Race and Union Status in the Wage Regression

Before:

$$LWAGE \rightarrow C, EDUC, EXPER$$

# Wage Regression on Education and Experience

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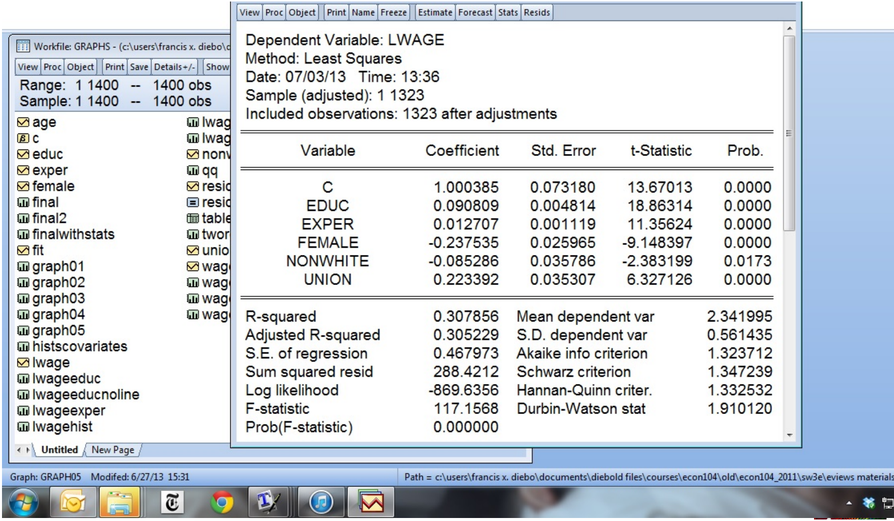
|---|---|---|---|
| R-squared | 0.232224 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.231061 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.492318 | Akaike info criterion | 1.422881 |
| Sum squared resid | 319.9376 | Schwarz criterion | 1.434644 |
| Log likelihood | -938.2358 | Hannan-Quinn criter. | 1.427291 |
| F-statistic | 199.6260 | Durbin-Watson stat | 1.926045 |
| Prob(F-statistic) | 0.000000 | | |

# Controlling for Sex, Race and Union Status in the Wage Regression

Now:

$$LWAGE \rightarrow C, EDUC, EXPER, FEMALE, NONWHITE, UNION$$

# Wage Regression on Education, Experience and Group Dummies

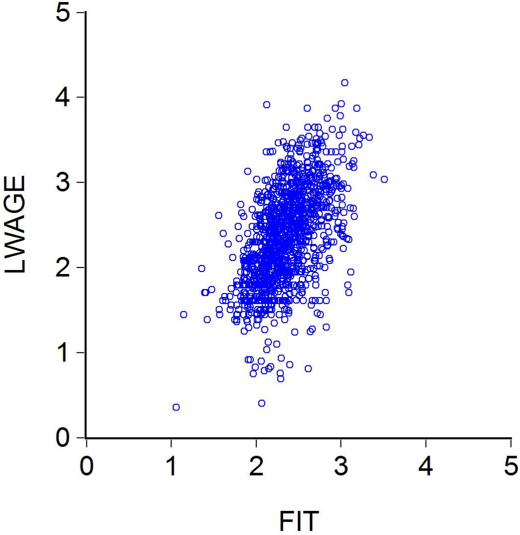



Dependent Variable: LWAGE
Method: Least Squares
Date: 07/03/13 Time: 13:36
Sample (adjusted): 1 1323
Included observations: 1323 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 1.000385 | 0.073180 | 13.67013 | 0.0000 |
| EDUC | 0.090809 | 0.004814 | 18.86314 | 0.0000 |
| EXPER | 0.012707 | 0.001119 | 11.35624 | 0.0000 |
| FEMALE | -0.237535 | 0.025965 | -9.148397 | 0.0000 |
| NONWHITE | -0.085286 | 0.035786 | -2.383199 | 0.0173 |
| UNION | 0.223392 | 0.035307 | 6.327126 | 0.0000 |

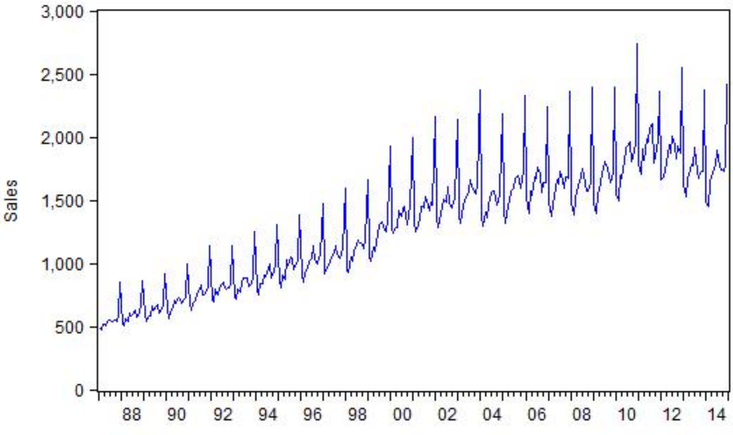
| | | | |
|---|---|---|---|
| R-squared | 0.307856 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.305229 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.467973 | Akaike info criterion | 1.323712 |
| Sum squared resid | 288.4212 | Schwarz criterion | 1.347239 |
| Log likelihood | -869.6356 | Hannan-Quinn criter. | 1.332532 |
| F-statistic | 117.1568 | Durbin-Watson stat | 1.910120 |
| Prob(F-statistic) | 0.000000 | | |

# Residual Scatter from Wage Regression on Education, Experience and Group Dummies

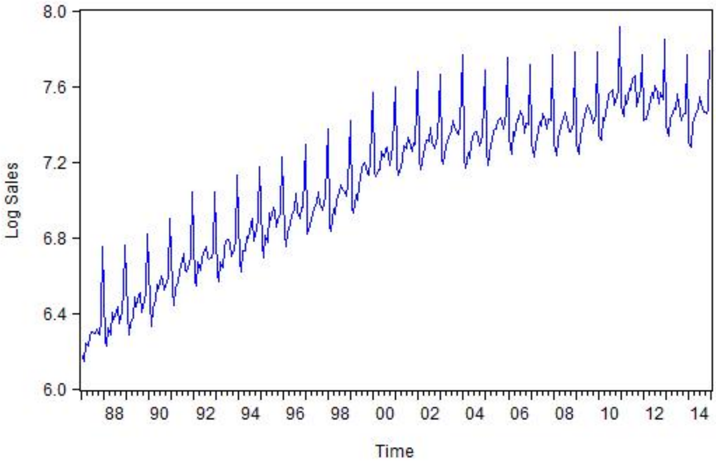


# Indicator Variables in Time Series: Trend and Seasonality

# Liquor Sales




# Log Liquor Sales



# Linear Deterministic Trend

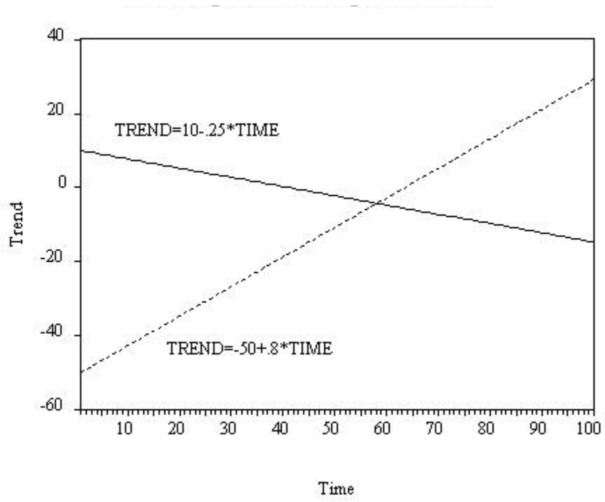
$$Trend_t = \beta_1 + \beta_2 TIME_t$$

where $TIME_t = t$

Simply run the least squares regression $y \to c, TIME$, where

$$TIME = \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ T-1 \\ T \end{pmatrix}$$

# Various Linear Trends



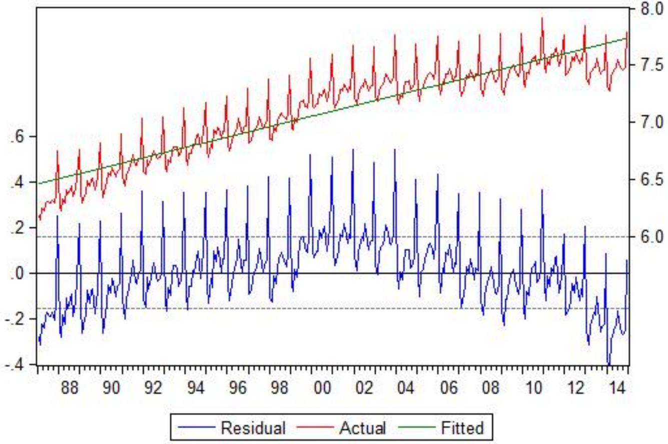
# Linear Trend Estimation

Method: Least Squares
Date: 08/08/13 Time: 08:53
Sample: 1987M01 2014M12
Included observations: 336

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 6.454290 | 0.017468 | 369.4834 | 0.0000 |
| TIME | 0.003809 | 8.98E-05 | 42.39935 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.843318 | Mean dependent var | 7.096188 |
| Adjusted R-squared | 0.842849 | S.D. dependent var | 0.402962 |
| S.E. of regression | 0.159743 | Akaike info criterion | -0.824561 |
| Sum squared resid | 8.523001 | Schwarz criterion | -0.801840 |
| Log likelihood | 140.5262 | Hannan-Quinn criter. | -0.815504 |
| F-statistic | 1797.705 | Durbin-Watson stat | 1.078573 |
| Prob(F-statistic) | 0.000000 | | |

# Residual Plot



# Deterministic Seasonality

$$Seasonal_t = \sum_{i=1}^{s} \beta_i SEAS_{it} \quad \text{(s seasons per year)}$$

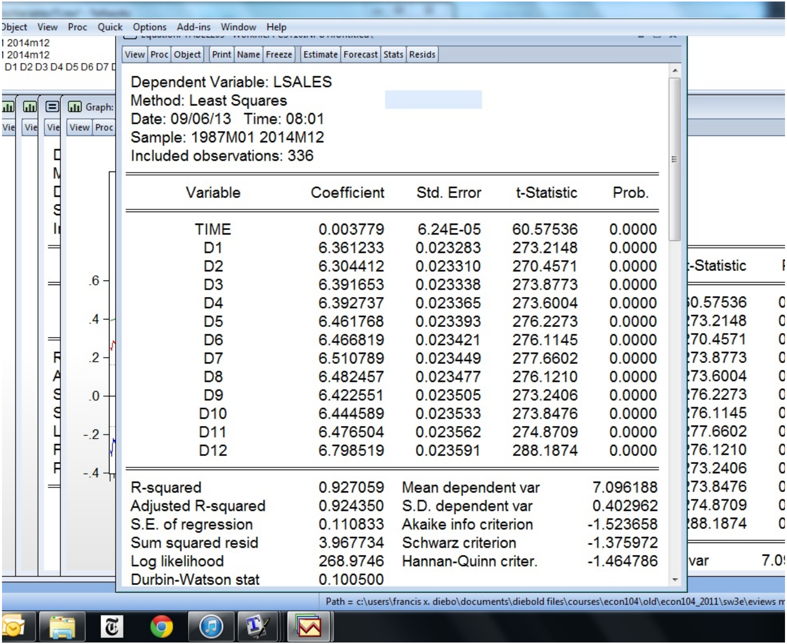
$$where\ SEAS_{it} = \begin{cases} 1 \text{ if observation } t \text{ falls in season } i \\ 0 \text{ otherwise} \end{cases}$$

Simply run the least squares regression $y \rightarrow SEAS_1, ..., SEAS_s$
(or blend: $y \rightarrow TIME, SEAS_1, ..., SEAS_s$)

where (e.g., in quarterly data case, assuming Q1 start and Q4 end):

$$SEAS_1 = (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, ..., 0)'$$
$$SEAS_2 = (0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, ..., 0)'$$
$$SEAS_3 = (0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, ..., 0)'$$
$$SEAS_4 = (0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, ..., 1)'.$$

# Linear Trend with Seasonal Dummies



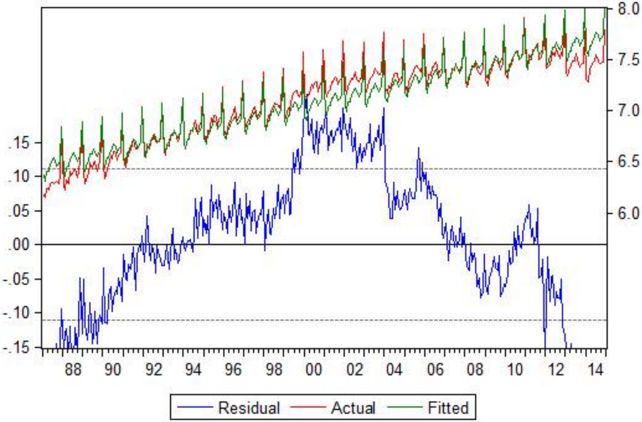
Dependent Variable: LSALES
Method: Least Squares
Date: 09/06/13 Time: 08:01
Sample: 1987M01 2014M12
Included observations: 336

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 0.003779 | 6.24E-05 | 60.57536 | 0.0000 |
| D1 | 6.361233 | 0.023283 | 273.2148 | 0.0000 |
| D2 | 6.304412 | 0.023310 | 270.4571 | 0.0000 |
| D3 | 6.391653 | 0.023338 | 273.8773 | 0.0000 |
| D4 | 6.392737 | 0.023365 | 273.6004 | 0.0000 |
| D5 | 6.461768 | 0.023393 | 276.2273 | 0.0000 |
| D6 | 6.466819 | 0.023421 | 276.1145 | 0.0000 |
| D7 | 6.510789 | 0.023449 | 277.6602 | 0.0000 |
| D8 | 6.482457 | 0.023477 | 276.1210 | 0.0000 |
| D9 | 6.422551 | 0.023505 | 273.2406 | 0.0000 |
| D10 | 6.444589 | 0.023533 | 273.8476 | 0.0000 |
| D11 | 6.476504 | 0.023562 | 274.8709 | 0.0000 |
| D12 | 6.798519 | 0.023591 | 288.1874 | 0.0000 |

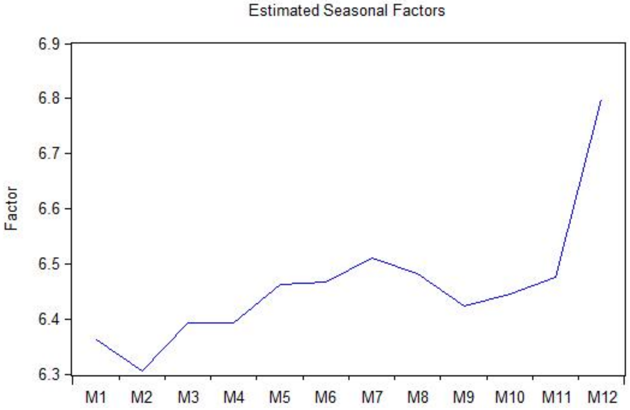
| | | | |
|---|---|---|---|
| R-squared | 0.927059 | Mean dependent var | 7.096188 |
| Adjusted R-squared | 0.924350 | S.D. dependent var | 0.402962 |
| S.E. of regression | 0.110833 | Akaike info criterion | -1.523658 |
| Sum squared resid | 3.967734 | Schwarz criterion | -1.375972 |
| Log likelihood | 268.9746 | Hannan-Quinn criter. | -1.464786 |
| Durbin-Watson stat | 0.100500 | | |



# Residual Plot

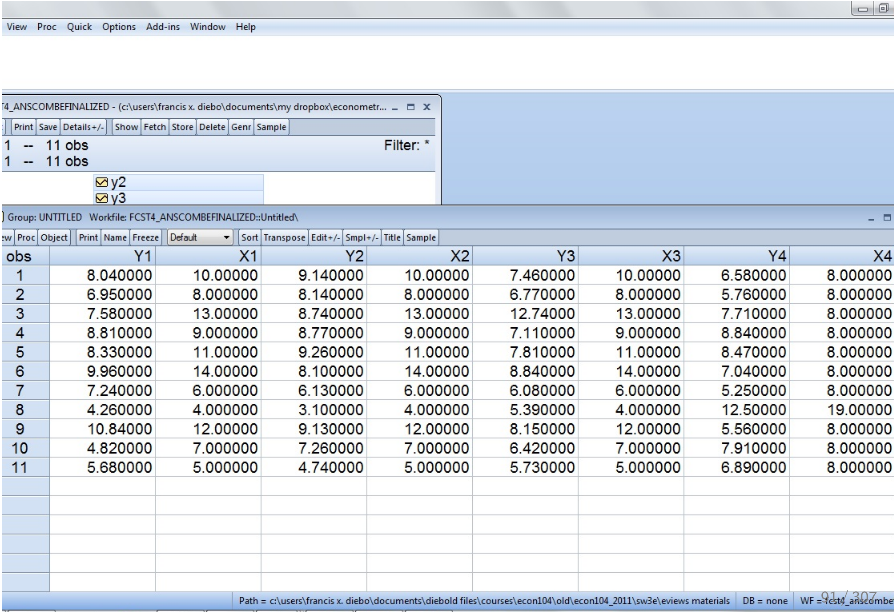



# Seasonal Pattern



# Nonlinearity in Cross Sections

# Anscombe's Quartet



| obs | Y1 | X1 | Y2 | X2 | Y3 | X3 | Y4 | X4 |
|---|---|---|---|---|---|---|---|---|
| 1 | 8.040000 | 10.00000 | 9.140000 | 10.00000 | 7.460000 | 10.00000 | 6.580000 | 8.000000 |
| 2 | 6.950000 | 8.000000 | 8.140000 | 8.000000 | 6.770000 | 8.000000 | 5.760000 | 8.000000 |
| 3 | 7.580000 | 13.00000 | 8.740000 | 13.00000 | 12.74000 | 13.00000 | 7.710000 | 8.000000 |
| 4 | 8.810000 | 9.000000 | 8.770000 | 9.000000 | 7.110000 | 9.000000 | 8.840000 | 8.000000 |
| 5 | 8.330000 | 11.00000 | 9.260000 | 11.00000 | 7.810000 | 11.00000 | 8.470000 | 8.000000 |
| 6 | 9.960000 | 14.00000 | 8.100000 | 14.00000 | 8.840000 | 14.00000 | 7.040000 | 8.000000 |
| 7 | 7.240000 | 6.000000 | 6.130000 | 6.000000 | 6.080000 | 6.000000 | 5.250000 | 8.000000 |
| 8 | 4.260000 | 4.000000 | 3.100000 | 4.000000 | 5.390000 | 4.000000 | 12.50000 | 19.00000 |
| 9 | 10.84000 | 12.00000 | 9.130000 | 12.00000 | 8.150000 | 12.00000 | 5.560000 | 8.000000 |
| 10 | 4.820000 | 7.000000 | 7.260000 | 7.000000 | 6.420000 | 7.000000 | 7.910000 | 8.000000 |
| 11 | 5.680000 | 5.000000 | 4.740000 | 5.000000 | 5.730000 | 5.000000 | 6.890000 | 8.000000 |

# Anscombe's Quartet: Regressions

LS // Dependent Variable is Y1

| Variable | Coefficient | Std Error | T-Statistic |
|---|---|---|---|
| C | 3.00 | 1.12 | 2.67 |
| X1 | 0.50 | 0.12 | 4.24 |

R-squared 0.67 S.E. of regression 1.24

LS // Dependent Variable is Y2

| Variable | Coefficient | Std Error | T-Statistic |
|---|---|---|---|
| C | 3.00 | 1.12 | 2.67 |
| X2 | 0.50 | 0.12 | 4.24 |

R-squared 0.67 S.E. of regression 1.24

LS // Dependent Variable is Y3

| Variable | Coefficient | Std Error | T-Statistic |
|---|---|---|---|
| C | 3.00 | 1.12 | 2.67 |
| X3 | 0.50 | 0.12 | 4.24 |

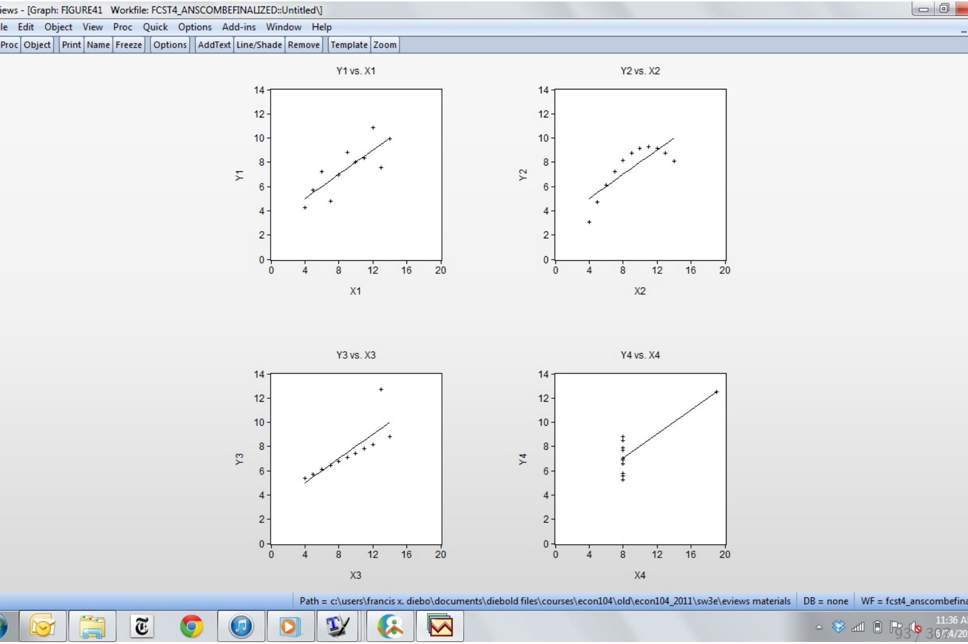
R-squared 0.67 S.E. of regression 1.24

LS // Dependent Variable is Y4

| Variable | Coefficient | Std Error | T-Statistic |
|---|---|---|---|
| C | 3.00 | 1.12 | 2.67 |
| X4 | 0.50 | 0.12 | 4.24 |

R-squared 0.67 S.E. of regression 1.24

# Anscombe's Quartet: Graphics



# Parametric and Nonparametric Nonlinearity...

...and the gray area in between.

# Log-Log Regression

$$lny_t = \beta_1 + \beta_2 lnx_t + \varepsilon_t$$

Example: Cobb-Douglas production function

$$y_t = AL_t^{\alpha} K_t^{\beta} exp(\varepsilon_t)$$

$$lny_t = lnA + \alpha lnL_t + \beta lnK_t + \varepsilon_t$$

For close $y_t$ and $x_t$, $(ln\, y_t - ln\, x_t)$ is approximately the percent difference between $y_t$ and $x_t$. Hence the coefficients in log-log regressions give the expected percent change in $E(y_t|x_t)$ for a one-percent change in $x_t$, the *elasticity of* $y_t$ *with respect to* $x_t$.

# Log-Lin Regression

$$lny_t = \beta x_t + \varepsilon$$

Example: Exponential growth

$$y_t = Ae^{rt}$$

$$lny_t = lnA + rt$$

The growth rate $r$ gives the approximate percent change in $E(y_t|t)$ for a one-unit change in time

Example: LWAGE regression!

# Box-Cox Regression

$$B(y_t) = \beta_1 + \beta_2 x_t + \varepsilon_t$$

where

$$B(y_t) = \frac{y_t^{\lambda} - 1}{\lambda}$$

Because

$$lim_{\lambda \to 0} \left( \frac{y^{\lambda} - 1}{\lambda} \right) = ln(y_t),$$

the Box-Cox model corresponds to the log-lin model in the special case of $\lambda = 0$.

# Generalized Linear Model

$$G(y_t) = \beta_1 + \beta_2 x_t + \varepsilon_t,$$

Wide classes of link functions $G$ can be entertained. Log-lin regression, for example, emerges when $G(y_t) = ln(y_t)$, and Box-Cox regression emerges when $G(y_t) = \frac{y_t^{\lambda} - 1}{\lambda}$.

# Intrinsically Non-Linear Models

One example is the logistic model,

$$y = \frac{1}{a + br^x}$$

$$(0 < r < 1)$$

– No way to transform to linearity

– Use non-linear least squares (NLS)

– Under the remaining FIC (that is, dropping only linearity), $\hat{\beta}_{NLS}$ has a sampling distribution similar to that of $\hat{\beta}_{LS}$ under the FIC

# Series Expansions

Really no such thing as an intrinsically non-linear model...

In the bivariate case we can think of the relationship as

$$y_t = g(x_t, \varepsilon_t)$$

or slightly less generally as

$$y_t = f(x_t) + \varepsilon_t$$

# Taylor

First consider Taylor series expansions of $f(x_t)$.
The linear (first-order) approximation is

$$f(x_t) \approx \beta_1 + \beta_2 x,$$

and the quadratic (second-order) approximation is

$$f(x_t) \approx \beta_1 + \beta_2 x_t + \beta_3 x_t^2.$$

In the multiple regression case, Taylor approximations also involve interaction terms. Consider, for example, $f(x_t, z_t)$:

$$f(x_t, z_t) \approx \beta_1 + \beta_2 x_t + \beta_3 z_t + \beta_4 x_t^2 + \beta_5 z_t^2 + \beta_6 x_t z_t + \ldots.$$

– Equally relevant for dummy variables: "interactions"

# Fourier

$$f(x_t) \approx \beta_1 + \beta_2 sin(x_t) + \beta_3 cos(x_t) + \beta_4 sin(2x_t) + \beta_5 cos(2x_t) + ...$$

– One can also mix Taylor and Fourier approximations by regressing not only on powers and cross products ("Taylor terms"), but also on various sines and cosines ("Fourier terms"). Mixing may facilitate parsimony.

# A Key Insight

The ultimate point is that so-called "intrinsically non-linear" models are themselves linear when viewed from the series-expansion perspective. In principle, of course, an infinite number of series terms are required, but in practice nonlinearity is often quite gentle (e.g., quadratic) so that only a few series terms are required.

– So non-linearity is in some sense
really an omitted-variables problem

# Testing for Non-Linearity I: $t$ and $F$ Tests

Just test for omitted series expansion terms!

# Testing for Non-Linearity II: RESET

Run:

$$y_t \to c, X_t$$

and obtain the fitted values $\hat{y}_t$.

Then run

$$y_t \to c, X_t, \hat{y}_t^2, ..., \hat{y}_t^m.$$

Note that the powers of $\hat{y}_t$ are linear combinations of powers and cross products of the $X$ variables. No need to include the first power of $\hat{y}_t$, because that would be redundant with the included $X$ variables. Instead we include powers $\hat{y}_t^2$, $\hat{y}_t^3$, ... Typically a small $m$ is adequate. Significance of the included set of powers of $\hat{y}_t$ can be checked using an $F$ test.

# Basic Wage Regression

Dependent Variable: LWAGE
Method: Least Squares
Date: 07/03/13 Time: 13:36
Sample (adjusted): 1 1323
Included observations: 1323 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 1.000385 | 0.073180 | 13.67013 | 0.0000 |
| EDUC | 0.090809 | 0.004814 | 18.86314 | 0.0000 |
| EXPER | 0.012707 | 0.001119 | 11.35624 | 0.0000 |
| FEMALE | -0.237535 | 0.025965 | -9.148397 | 0.0000 |
| NONWHITE | -0.085286 | 0.035786 | -2.383199 | 0.0173 |
| UNION | 0.223392 | 0.035307 | 6.327126 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.307856 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.305229 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.467973 | Akaike info criterion | 1.323712 |
| Sum squared resid | 288.4212 | Schwarz criterion | 1.347239 |
| Log likelihood | -869.6356 | Hannan-Quinn criter. | 1.332532 |
| F-statistic | 117.1568 | Durbin-Watson stat | 1.910120 |
| Prob(F-statistic) | 0.000000 | | |

# Quadratic Wage Regression

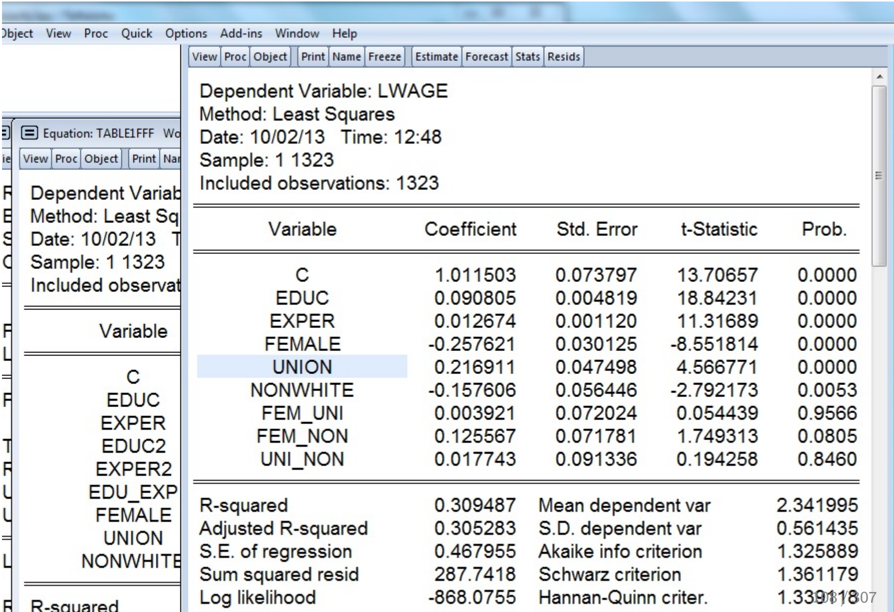
Dependent Variable: LWAGE
Method: Least Squares
Date: 10/02/13 Time: 12:37
Sample: 1 1323
Included observations: 1323

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.473236 | 0.240586 | 1.967017 | 0.0494 |
| EDUC | 0.109673 | 0.028918 | 3.792608 | 0.0002 |
| EXPER | 0.064422 | 0.007652 | 8.419060 | 0.0000 |
| EDUC2 | 0.000501 | 0.000895 | 0.559994 | 0.5756 |
| EXPER2 | -0.000705 | 8.86E-05 | -7.962263 | 0.0000 |
| EDU_EXP | -0.001789 | 0.000429 | -4.173423 | 0.0000 |
| FEMALE | -0.237696 | 0.025506 | -9.319335 | 0.0000 |
| UNION | 0.202955 | 0.034569 | 5.870998 | 0.0000 |
| NONWHITE | -0.095028 | 0.034931 | -2.720476 | 0.0066 |

| | | | |
|---|---|---|---|
| R-squared | 0.343072 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.339073 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.456433 | Akaike info criterion | 1.276028 |
| Sum squared resid | 273.7465 | Schwarz criterion | 1.311318 |
| Log likelihood | -835.0925 | Hannan-Quinn criter. | 1.289257 |
| F-statistic | 85.77745 | Durbin-Watson stat | 1.894409 |

Path = c:\users\francis x. diebold\documents\diebold files\books\econometrics\dataandsoftware\aaawagesmaster\

# Dummy Interactions?

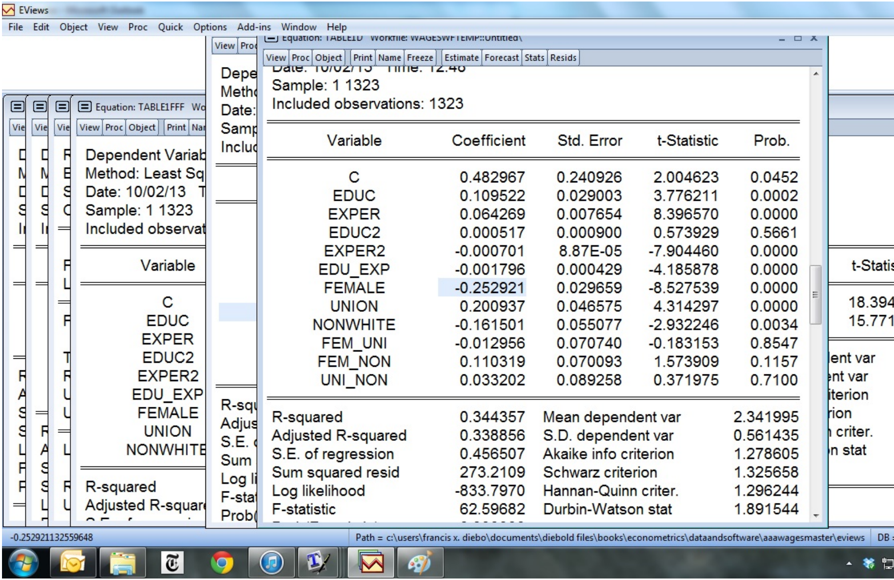



Dependent Variable: LWAGE
Method: Least Squares
Date: 10/02/13 Time: 12:48
Sample: 1 1323
Included observations: 1323

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 1.011503 | 0.073797 | 13.70657 | 0.0000 |
| EDUC | 0.090805 | 0.004819 | 18.84231 | 0.0000 |
| EXPER | 0.012674 | 0.001120 | 11.31689 | 0.0000 |
| FEMALE | -0.257621 | 0.030125 | -8.551814 | 0.0000 |
| UNION | 0.216911 | 0.047498 | 4.566771 | 0.0000 |
| NONWHITE | -0.157606 | 0.056446 | -2.792173 | 0.0053 |
| FEM_UNI | 0.003921 | 0.072024 | 0.054439 | 0.9566 |
| FEM_NON | 0.125567 | 0.071781 | 1.749313 | 0.0805 |
| UNI_NON | 0.017743 | 0.091336 | 0.194258 | 0.8460 |

| | | | |
|---|---|---|---|
| R-squared | 0.309487 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.305283 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.467955 | Akaike info criterion | 1.325889 |
| Sum squared resid | 287.7418 | Schwarz criterion | 1.361179 |
| Log likelihood | -868.0755 | Hannan-Quinn criter. | 1.33918 |

# Everything

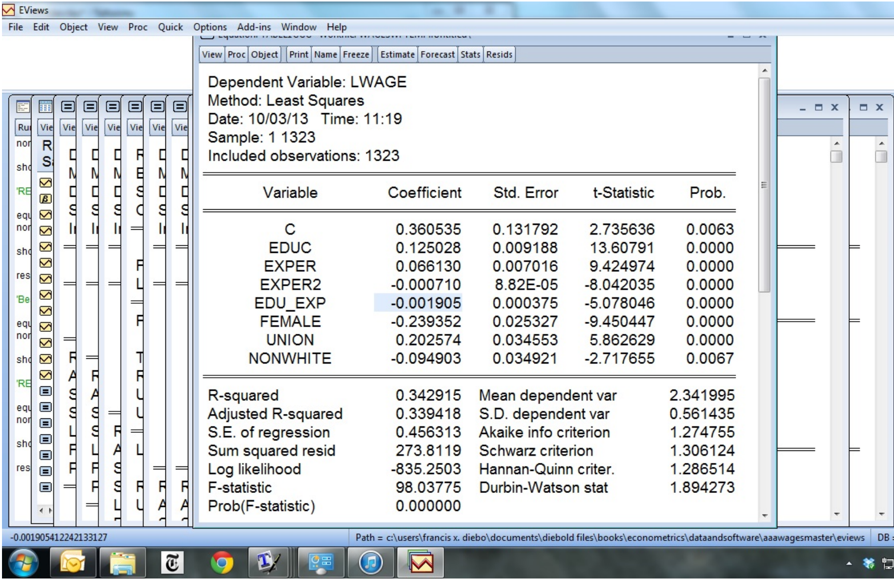


Sample: 1 1323
Included observations: 1323

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.482967 | 0.240926 | 2.004623 | 0.0452 |
| EDUC | 0.109522 | 0.029003 | 3.776211 | 0.0002 |
| EXPER | 0.064269 | 0.007654 | 8.396570 | 0.0000 |
| EDUC2 | 0.000517 | 0.000900 | 0.573929 | 0.5661 |
| EXPER2 | -0.000701 | 8.87E-05 | -7.904460 | 0.0000 |
| EDU_EXP | -0.001796 | 0.000429 | -4.185878 | 0.0000 |
| FEMALE | -0.252921 | 0.029659 | -8.527539 | 0.0000 |
| UNION | 0.200937 | 0.046575 | 4.314297 | 0.0000 |
| NONWHITE | -0.161501 | 0.055077 | -2.932246 | 0.0034 |
| FEM_UNI | -0.012956 | 0.070740 | -0.183153 | 0.8547 |
| FEM_NON | 0.110319 | 0.070093 | 1.573909 | 0.1157 |
| UNI_NON | 0.033202 | 0.089258 | 0.371975 | 0.7100 |

| | | | |
|---|---|---|---|
| R-squared | 0.344357 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.338856 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.456507 | Akaike info criterion | 1.278605 |
| Sum squared resid | 273.2109 | Schwarz criterion | 1.325658 |
| Log likelihood | -833.7970 | Hannan-Quinn criter. | 1.296244 |
| F-statistic | 62.59682 | Durbin-Watson stat | 1.891544 |

# So Drop Dummy Interactions and Tighten the Rest



Dependent Variable: LWAGE
Method: Least Squares
Date: 10/03/13 Time: 11:19
Sample: 1 1323
Included observations: 1323

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.360535 | 0.131792 | 2.735636 | 0.0063 |
| EDUC | 0.125028 | 0.009188 | 13.60791 | 0.0000 |
| EXPER | 0.066130 | 0.007016 | 9.424974 | 0.0000 |
| EXPER2 | -0.000710 | 8.82E-05 | -8.042035 | 0.0000 |
| EDU_EXP | -0.001905 | 0.000375 | -5.078046 | 0.0000 |
| FEMALE | -0.239352 | 0.025327 | -9.450447 | 0.0000 |
| UNION | 0.202574 | 0.034553 | 5.862629 | 0.0000 |
| NONWHITE | -0.094903 | 0.034921 | -2.717655 | 0.0067 |

| | | | |
|---|---|---|---|
| R-squared | 0.342915 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.339418 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.456313 | Akaike info criterion | 1.274755 |
| Sum squared resid | 273.8119 | Schwarz criterion | 1.306124 |
| Log likelihood | -835.2503 | Hannan-Quinn criter. | 1.286514 |
| F-statistic | 98.03775 | Durbin-Watson stat | 1.894273 |
| Prob(F-statistic) | 0.000000 | | |

# Nonlinearity in Time Series

# Non-Linear Trend: Exponential (Log-Linear)

$$Trend_t = \beta_1 e^{\beta_2 TIME_t}$$

$$\ln(Trend_t) = \ln(\beta_1) + \beta_2 TIME_t$$

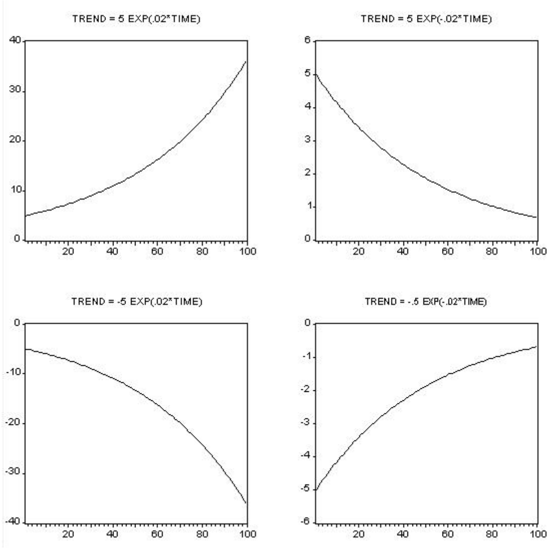



Figure: Various Exponential Trends

# Non-Linear Trend: Quadratic

Allow for gentle curvature by including *TIME and TIME*$^2$:

$$Trend_t = \beta_1 + \beta_2 TIME_t + \beta_3 TIME_t^2$$

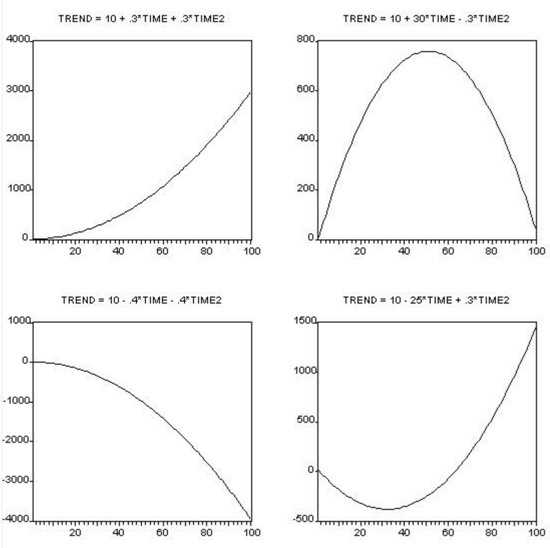



Figure: Various Quadratic Trends

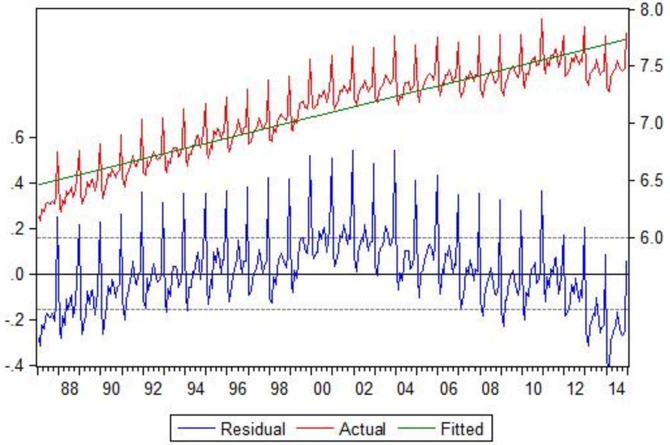
# Recall Log-Linear Liquor Sales Trend Estimation

Method: Least Squares
Date: 08/08/13 Time: 08:53
Sample: 1987M01 2014M12
Included observations: 336

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 6.454290 | 0.017468 | 369.4834 | 0.0000 |
| TIME | 0.003809 | 8.98E-05 | 42.39935 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.843318 | Mean dependent var | 7.096188 |
| Adjusted R-squared | 0.842849 | S.D. dependent var | 0.402962 |
| S.E. of regression | 0.159743 | Akaike info criterion | -0.824561 |
| Sum squared resid | 8.523001 | Schwarz criterion | -0.801840 |
| Log likelihood | 140.5262 | Hannan-Quinn criter. | -0.815504 |
| F-statistic | 1797.705 | Durbin-Watson stat | 1.078573 |
| Prob(F-statistic) | 0.000000 | | |

# Residual Plot



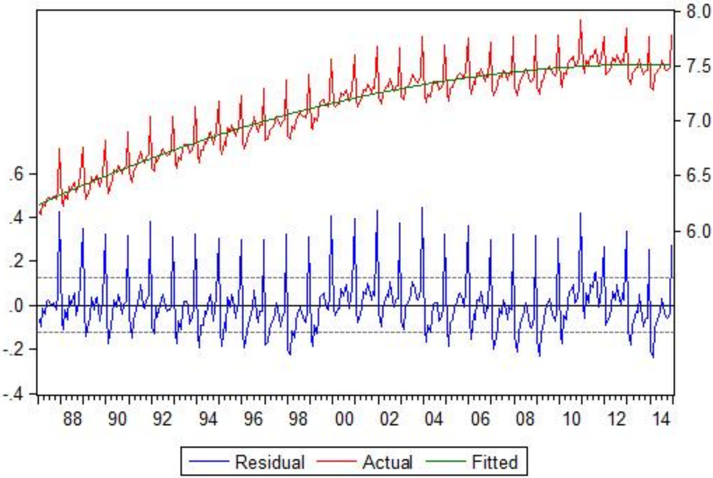
# Log-Quadratic Liquor Sales Trend Estimation

Dependent Variable: LSALES
Method: Least Squares
Date: 08/08/13 Time: 08:53
Sample: 1987M01 2014M12
Included observations: 336

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 6.231269 | 0.020653 | 301.7187 | 0.0000 |
| TIME | 0.007768 | 0.000283 | 27.44987 | 0.0000 |
| TIME2 | -1.17E-05 | 8.13E-07 | -14.44511 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.903676 | Mean dependent var | 7.096188 |
| Adjusted R-squared | 0.903097 | S.D. dependent var | 0.402962 |
| S.E. of regression | 0.125439 | Akaike info criterion | -1.305106 |
| Sum squared resid | 5.239733 | Schwarz criterion | -1.271025 |
| Log likelihood | 222.2579 | Hannan-Quinn criter. | -1.291521 |
| F-statistic | 1562.036 | Durbin-Watson stat | 1.754412 |
| Prob(F-statistic) | 0.000000 | | |

# Residual Plot



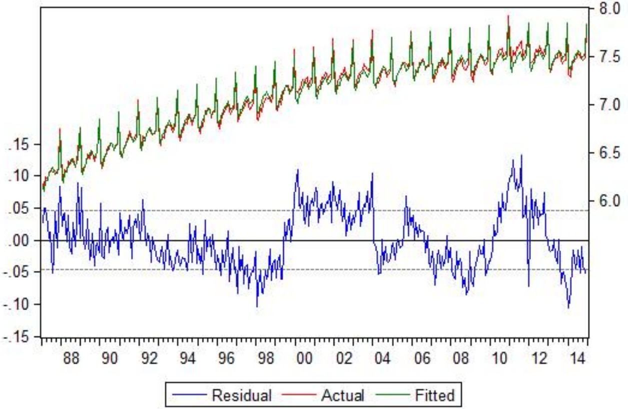
# Log-Quadratic Liquor Sales Trend Estimation with Seasonal Dummies

Dependent Variable: LSALES
Method: Least Squares
Date: 08/08/13 Time: 08:53
Sample: 1987M01 2014M12
Included observations: 336

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 0.007739 | 0.000104 | 74.49828 | 0.0000 |
| TIME2 | -1.18E-05 | 2.98E-07 | -39.36756 | 0.0000 |
| D1 | 6.138362 | 0.011207 | 547.7315 | 0.0000 |
| D2 | 6.081424 | 0.011218 | 542.1044 | 0.0000 |
| D3 | 6.168571 | 0.011229 | 549.3318 | 0.0000 |
| D4 | 6.169584 | 0.011240 | 548.8944 | 0.0000 |
| D5 | 6.238568 | 0.011251 | 554.5117 | 0.0000 |
| D6 | 6.243596 | 0.011261 | 554.4513 | 0.0000 |
| D7 | 6.287566 | 0.011271 | 557.8584 | 0.0000 |
| D8 | 6.259257 | 0.011281 | 554.8647 | 0.0000 |
| D9 | 6.199399 | 0.011290 | 549.0938 | 0.0000 |
| D10 | 6.221507 | 0.011300 | 550.5987 | 0.0000 |
| D11 | 6.253515 | 0.011309 | 552.9885 | 0.0000 |
| D12 | 6.575648 | 0.011317 | 581.0220 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.987452 | Mean dependent var | 7.096188 |
| Adjusted R-squared | 0.986946 | S.D. dependent var | 0.402962 |
| S.E. of regression | 0.046041 | Akaike info criterion | -3.277812 |
| Sum squared resid | 0.682555 | Schwarz criterion | -3.118766 |
| Log likelihood | 564.6725 | Hannan-Quinn criter. | -3.214412 |
| Durbin-Watson stat | 0.581383 | | |

# Residual Plot



# Moving-Average Trend and De-Trending

Two-sided moving average:

$$s_t = \frac{1}{2m+1} \sum_{i=-m}^{m} y_{t-i}$$

One-sided moving average:

$$s_t = \frac{1}{m+1} \sum_{i=0}^{m} y_{t-i}$$

One-sided weighted moving average:

$$s_t = \sum_{i=0}^{m} w_i y_{t-i}$$

# Hodrick-Prescott Trend and De-Trending

$$\min_{\{s_t\}_{t=1}^{T}} \left( \sum_{t=1}^{T} (y_t - s_t)^2 + \lambda \sum_{t=2}^{T-1} \left( (s_{t+1} - s_t) - (s_t - s_{t-1}) \right)^2 \right)$$

# More Problems

# Measurement Error

DGP:

$$y_t = \beta_1 + \beta_2 x_t + \varepsilon_t$$

Measurement:

$$x_t^m = x_t + v_t, \quad v_t \sim iid(0, \sigma^2)$$

We incorrectly run:

$$y \rightarrow c, x^m$$

As $\sigma_v^2 \,/\, \sigma_x^2$ gets large, the regression is progressively less able to identify the true relationship. In the limit as $\sigma_v^2 \,/\, \sigma_x^2 \rightarrow \infty$, it is impossible. In any event, $\hat{\beta}_{LS}$ is biased toward zero, in small as well as large samples.

# Omitted Variables

DGP:

$$y_t = \beta_1 + \beta_2 z_t + \varepsilon_t$$

We incorrectly run:

$$y \rightarrow c, x$$

where $corr(x_t, z_t) > 0$.

Clearly we'll estimate a positive effect of $x$ on $y$, in large as well as small samples, even though it's completely spurious and would vanish if $z$ had been included in the regression. The positive bias arises because in our example we assumed that $corr(z_t, z_t) > 0$; in general the sign of the bias could go either way.

# Multicollinearity

Perfect Multicollinearity (e.g., dummy-variable trap):

– Drop a variable!

Imperfect Multicollinearity:

– Large $F$ and $R^2$, yet small $t$'s (large s.e.'s). Hard to parse effects of $x$'s on $y$, yet it's clear that there is an overall relationship.

– That's just the way life is. Not really a "problem."

– OLS is natural: orthogonal projection.

# Multicollinearity and Variance Inflation

$$var(\hat{\beta}_j) = f\left(\underbrace{\sigma^2}_{+}, \underbrace{\sigma^2_{x_j}}_{-}, \underbrace{R^2_j}_{+}\right)$$

where $R^2_j$ is regression of $x_j$ on all other regressors

$R^2_j$ affects $var(\hat{\beta}_j)$ as $(1 - R^2_j)^{-1}$
Hence as $R^2_j \rightarrow 1$ the variance inflation approaches infinity
($x_j$ completely redundant)

# Non-Normality and Outliers

– Distributional results

– Diagnostics

– Outliers

– Robust estimation

# Recall Sample Mean Under *iid* Normality

$\bar{y}$ is unbiased, consistent, normally distributed with variance $\sigma^2/T$, and indeed the minimum variance unbiased (MVUE) estimator.

We write:

$$\bar{y} \sim N\left(\mu, \frac{\sigma^2}{T}\right)$$

or equivalently

$$\sqrt{T}(\bar{y} - \mu) \sim N(0, \sigma^2)$$

# Recall Sample Mean Under *iid* (Less Normality)

$\bar{y}$ is unbiased, consistent, *asymptotically* normally distributed with variance $\sigma^2/T$, and best linear unbiased (BLUE).

We write:

$$\bar{y} \overset{a}{\sim} N\left(\mu, \frac{\sigma^2}{T}\right)$$

or more precisely, as $T \to \infty$,

$$\sqrt{T}(\bar{y} - \mu) \to_d N(0, \sigma^2)$$

# OLS Under FIC (Including Normality)

$\hat{\beta}_{LS}$ is unbiased, consistent, normally distributed with covariance matrix $\sigma^2(X'X)^{-1}$, and indeed MVUE.

We write:

$$\hat{\beta}_{LS} \sim N\left(\beta,\ \sigma^2(X'X)^{-1}\right)$$

or equivalently

$$\sqrt{T}(\hat{\beta} - \beta) \sim N\left(0, \sigma^2\left(\frac{X'X}{T}\right)^{-1}\right)$$

# OLS Under FIC (Less Normality)

$\hat{\beta}_{LS}$ is consistent, *asymptotically* normally distributed, and BLUE.

We write

$$\hat{\beta}_{LS} \overset{a}{\sim} N\left(\beta,\ \sigma^2 (X'X)^{-1}\right),$$

or more precisely, as $T \to \infty$,

$$\sqrt{T}(\hat{\beta}_{LS} - \beta) \to_d N\left(0,\ \sigma^2 \left(\frac{X'X}{T}\right)^{-1}\right)$$

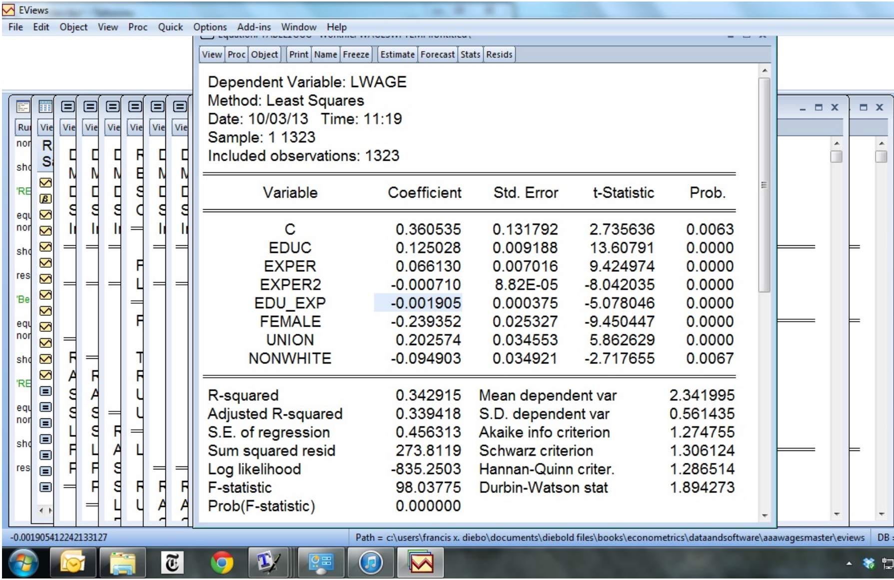
# Residual Normality Diagnostics

- Sample skewness and kurtosis, $\hat{S}$ and $\hat{K}$
- Jarque-Bera test. Under normality we have:

$$JB = \frac{T}{6}\left(\hat{S}^2 + \frac{1}{4}(\hat{K}-3)^2\right) \sim \chi^2_2$$

- More exotic: Outlier probabilities, tail indexes
- All can be done on observed data or residuals

# Recall Our "Final" Wage Regression

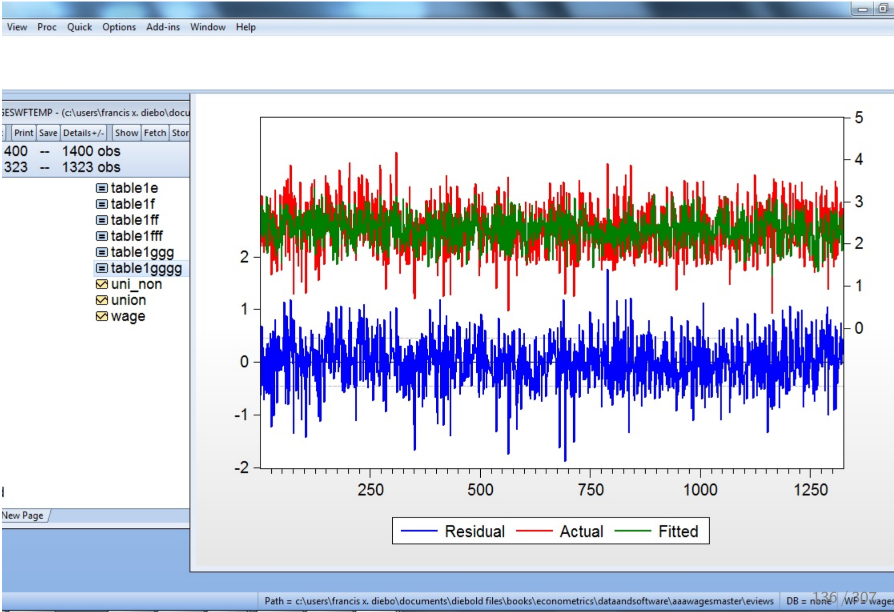


Dependent Variable: LWAGE
Method: Least Squares
Date: 10/03/13 Time: 11:19
Sample: 1 1323
Included observations: 1323

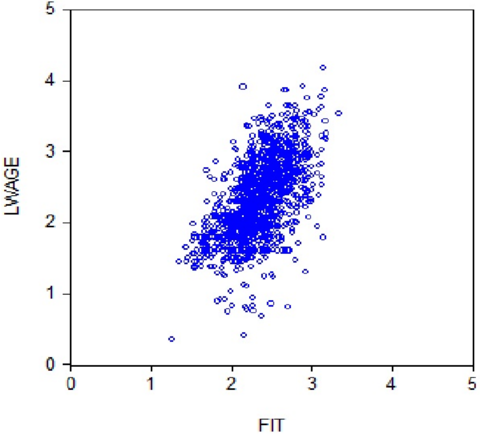
| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.360535 | 0.131792 | 2.735636 | 0.0063 |
| EDUC | 0.125028 | 0.009188 | 13.60791 | 0.0000 |
| EXPER | 0.066130 | 0.007016 | 9.424974 | 0.0000 |
| EXPER2 | -0.000710 | 8.82E-05 | -8.042035 | 0.0000 |
| EDU_EXP | -0.001905 | 0.000375 | -5.078046 | 0.0000 |
| FEMALE | -0.239352 | 0.025327 | -9.450447 | 0.0000 |
| UNION | 0.202574 | 0.034553 | 5.862629 | 0.0000 |
| NONWHITE | -0.094903 | 0.034921 | -2.717655 | 0.0067 |

| | | | |
|---|---|---|---|
| R-squared | 0.342915 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.339418 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.456313 | Akaike info criterion | 1.274755 |
| Sum squared resid | 273.8119 | Schwarz criterion | 1.306124 |
| Log likelihood | -835.2503 | Hannan-Quinn criter. | 1.286514 |
| F-statistic | 98.03775 | Durbin-Watson stat | 1.894273 |
| Prob(F-statistic) | 0.000000 | | |

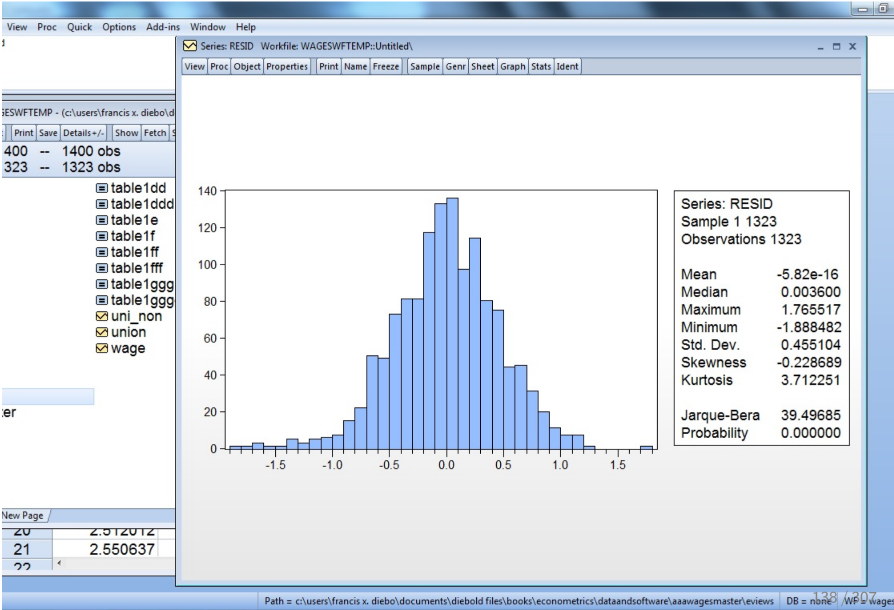
# Residual Plot



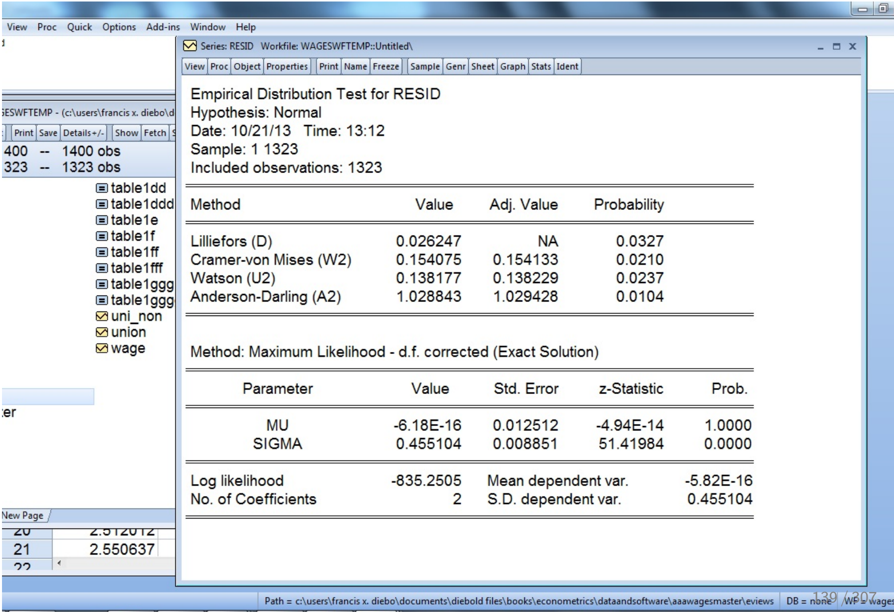
# Residual Scatter



# Residual Histogram and Statistics



# More Residual Normality Tests



Series: RESID Workfile: WAGESWFTEMP::Untitled\

Empirical Distribution Test for RESID
Hypothesis: Normal
Date: 10/21/13 Time: 13:12
Sample: 1 1323
Included observations: 1323

| Method | Value | Adj. Value | Probability |
|---|---|---|---|
| Lilliefors (D) | 0.026247 | NA | 0.0327 |
| Cramer-von Mises (W2) | 0.154075 | 0.154133 | 0.0210 |
| Watson (U2) | 0.138177 | 0.138229 | 0.0237 |
| Anderson-Darling (A2) | 1.028843 | 1.029428 | 0.0104 |

Method: Maximum Likelihood - d.f. corrected (Exact Solution)

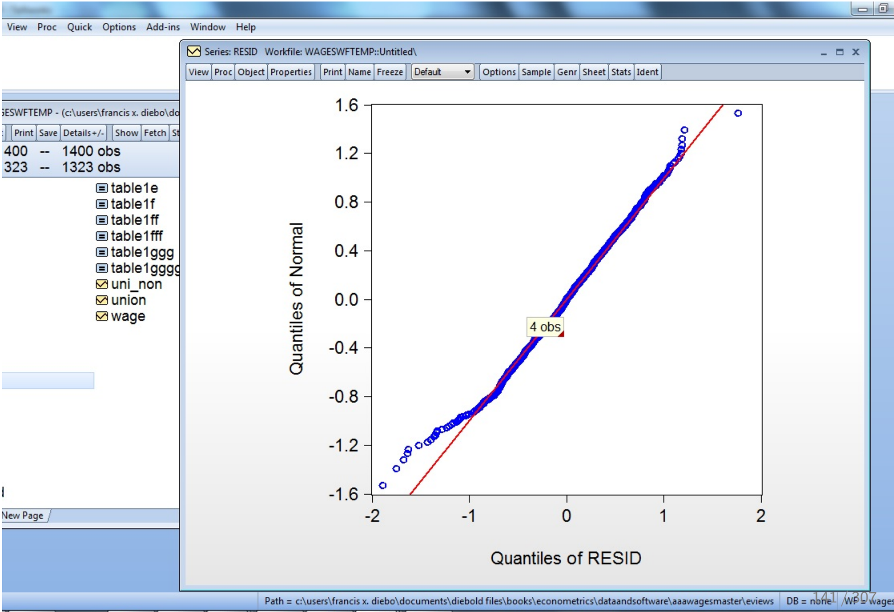
| Parameter | Value | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| MU | -6.18E-16 | 0.012512 | -4.94E-14 | 1.0000 |
| SIGMA | 0.455104 | 0.008851 | 51.41984 | 0.0000 |

| | | | |
|---|---|---|---|
| Log likelihood | -835.2505 | Mean dependent var. | -5.82E-16 |
| No. of Coefficients | 2 | S.D. dependent var. | 0.455104 |

# Residual QQ Plots

- We introduced histograms earlier...
- ...but if interest centers on the *tails* of distributions, QQ plots often provide sharper insight as to the agreement or divergence between the actual and reference distributions
- QQ plot is simply a plot of the quantiles of the standardized data against the quantiles of a standardized reference distribution (e.g., normal)
- If the distributions match, the QQ plot is the 45 degree line
- To the extent that the QQ plot does not match the 45 degree line, the nature of the divergence can be very informative, as for example in indicating fat tails
- Can be done on observed data or residuals

# Wage Regression Residual QQ Plot



# Outlier Detection and Robust Estimation

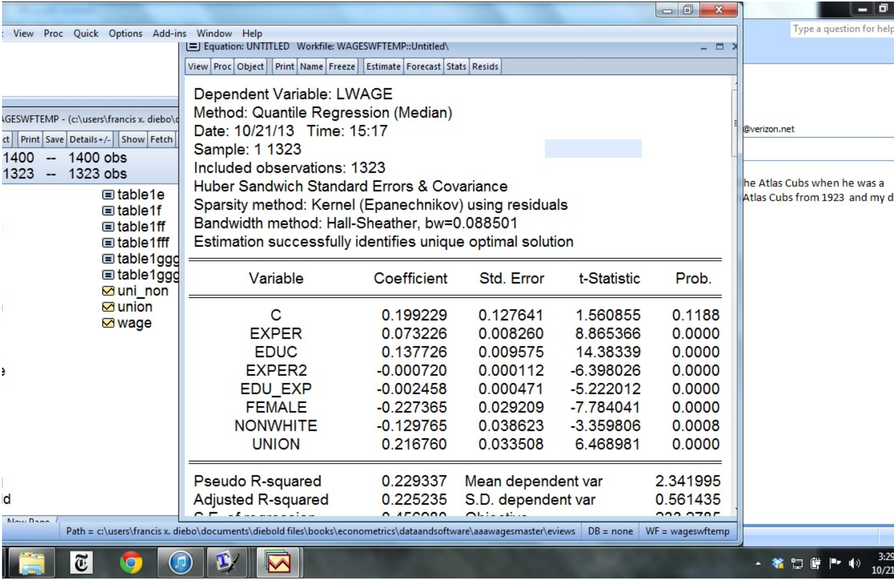
- Data scatterplots
- Residual plots and scatterplots
- "Leave-one-out" plots:

$$\left(\hat{\beta}_k - \hat{\beta}_k(-t)\right),\ t = 1, \ldots T\ (k = 1, \ldots, K)$$

- Robust estimation: LAD

$$\min_{\beta} \sum_{t=1}^{T} |y_t - \beta_1 - \beta_2 x_{2t} - \ldots - \beta_K x_{Kt}|$$

# Wage Regression LAD Estimation

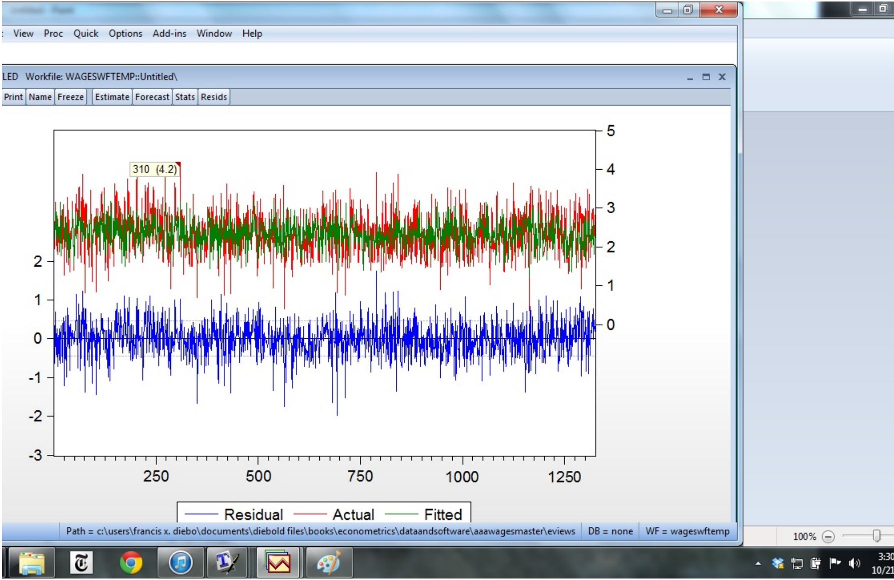


Dependent Variable: LWAGE
Method: Quantile Regression (Median)
Date: 10/21/13 Time: 15:17
Sample: 1 1323
Included observations: 1323
Huber Sandwich Standard Errors & Covariance
Sparsity method: Kernel (Epanechnikov) using residuals
Bandwidth method: Hall-Sheather, bw=0.088501
Estimation successfully identifies unique optimal solution

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.199229 | 0.127641 | 1.560855 | 0.1188 |
| EXPER | 0.073226 | 0.008260 | 8.865366 | 0.0000 |
| EDUC | 0.137726 | 0.009575 | 14.38339 | 0.0000 |
| EXPER2 | -0.000720 | 0.000112 | -6.398026 | 0.0000 |
| EDU_EXP | -0.002458 | 0.000471 | -5.222012 | 0.0000 |
| FEMALE | -0.227365 | 0.029209 | -7.784041 | 0.0000 |
| NONWHITE | -0.129765 | 0.038623 | -3.359806 | 0.0008 |
| UNION | 0.216760 | 0.033508 | 6.468981 | 0.0000 |

| | | | |
|---|---|---|---|
| Pseudo R-squared | 0.229337 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.225235 | S.D. dependent var | 0.561435 |

# Residual Plot



# Generalized Least Squares (GLS)

Consider the FIC except that we now let:

$$\varepsilon \sim N(\underline{0}, \sigma^2 \Omega)$$

The old case is $\Omega = I$, but things are very different when $\Omega \neq I$:
– OLS parameter estimates consistent but inefficient
(no longer MVUE or BLUE)

– OLS standard errors biased and inconsistent. Hence $t$ ratios do not have the $t$ distribution in finite samples and do not have the $N(0, 1)$ distribution asymptotically

The GLS estimator is:

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

Under the remaining full ideal conditions it is consistent, normally distributed with covariance matrix $\sigma^2(X'\Omega^{-1}X)^{-1}$, and MVUE:

$$\hat{\beta}_{GLS} \sim N\left(\beta,\ \sigma^2(X'\Omega^{-1}X)^{-1}\right).$$

# Heteroskedasticity in Cross-Section Regression

Homoskedasticity: variance of $\varepsilon_i$ is constant across $i$

Heteroskedasticity: variance of $\varepsilon_i$ is not constant across $i$

Relevant cross-sectional heteroskedasticity situation
(on which we focus for now):
$\varepsilon_i$ independent across $i$ but not identically distributed across $i$

$$\Omega = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N^2 \end{pmatrix}$$

– Can arise for many reasons
– Engel curve (e.g., food expenditure vs. income) is classic example

# Consequences

OLS inefficient (no longer MVUE or BLUE),
in finite samples and asymptotically

Standard errors biased and inconsistent.
Hence $t$ ratios do not have the $t$ distribution in finite samples
and do not have the $N(0, 1)$ distribution asymptotically

# Detection

- Graphical heteroskedasticity diagnostics

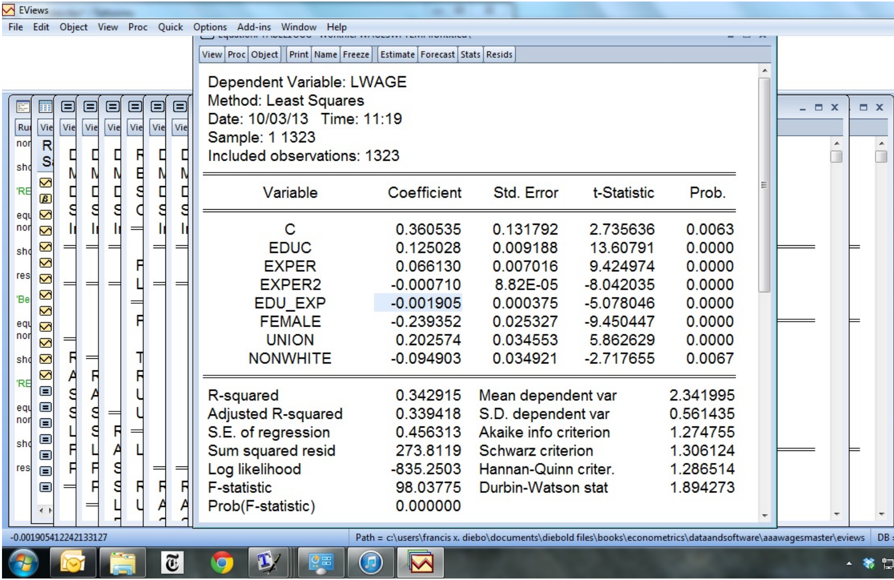
- Formal heteroskedasticity tests

# Graphical Diagnostics

Graph $e_i^2$ against $x_i$, for various regressors

Problem: Purely pairwise

# Recall Our “Final” Wage Regression

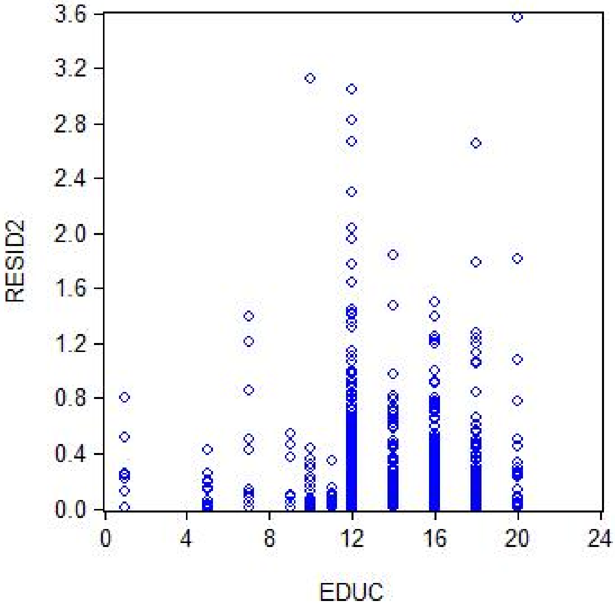


Dependent Variable: LWAGE
Method: Least Squares
Date: 10/03/13 Time: 11:19
Sample: 1 1323
Included observations: 1323

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.360535 | 0.131792 | 2.735636 | 0.0063 |
| EDUC | 0.125028 | 0.009188 | 13.60791 | 0.0000 |
| EXPER | 0.066130 | 0.007016 | 9.424974 | 0.0000 |
| EXPER2 | -0.000710 | 8.82E-05 | -8.042035 | 0.0000 |
| EDU_EXP | -0.001905 | 0.000375 | -5.078046 | 0.0000 |
| FEMALE | -0.239352 | 0.025327 | -9.450447 | 0.0000 |
| UNION | 0.202574 | 0.034553 | 5.862629 | 0.0000 |
| NONWHITE | -0.094903 | 0.034921 | -2.717655 | 0.0067 |

| | | | |
|---|---|---|---|
| R-squared | 0.342915 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.339418 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.456313 | Akaike info criterion | 1.274755 |
| Sum squared resid | 273.8119 | Schwarz criterion | 1.306124 |
| Log likelihood | -835.2503 | Hannan-Quinn criter. | 1.286514 |
| F-statistic | 98.03775 | Durbin-Watson stat | 1.894273 |
| Prob(F-statistic) | 0.000000 | | |

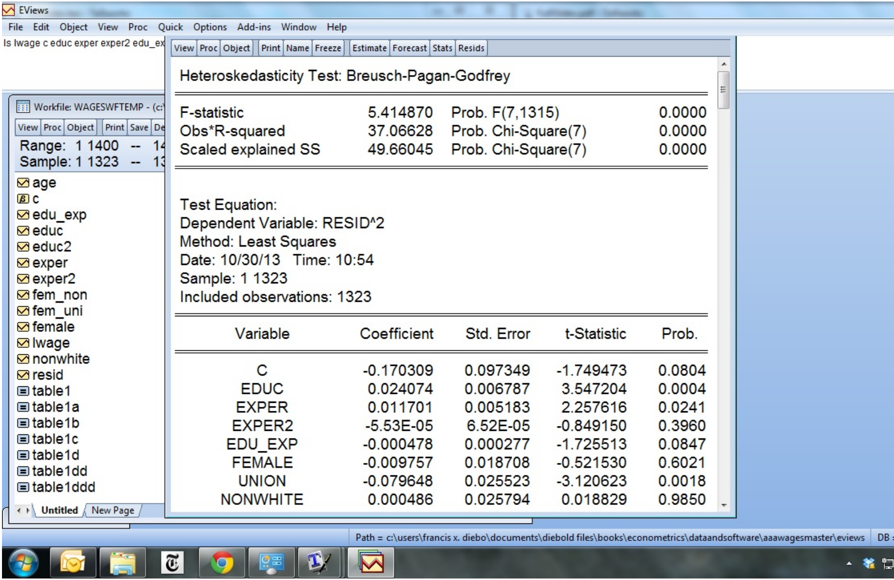
# Squared Residual vs. EDUC



# The Breusch-Godfrey-Pagan Test (BGP)

- Estimate the OLS regression, and obtain the squared residuals
- Regress the squared residuals on all regressors
- To test the null hypothesis of no relationship, examine $NR^2$ from this regression. In large samples $NR^2 \sim \chi^2$ under the null.

# BPG Test



Heteroskedasticity Test: Breusch-Pagan-Godfrey

| | | | |
|---|---|---|---|
| F-statistic | 5.414870 | Prob. F(7,1315) | 0.0000 |
| Obs*R-squared | 37.06628 | Prob. Chi-Square(7) | 0.0000 |
| Scaled explained SS | 49.66045 | Prob. Chi-Square(7) | 0.0000 |

Test Equation:
Dependent Variable: RESID^2
Method: Least Squares
Date: 10/30/13 Time: 10:54
Sample: 1 1323
Included observations: 1323

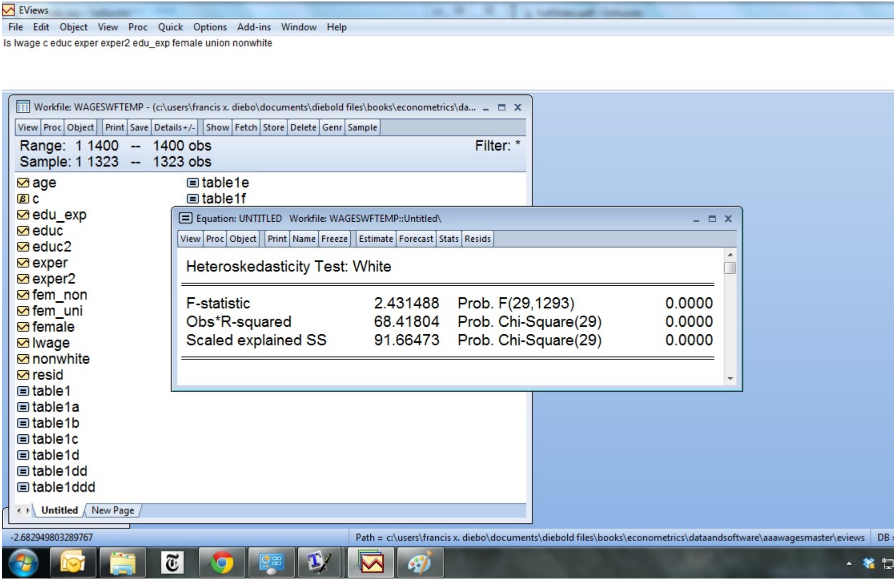
| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | -0.170309 | 0.097349 | -1.749473 | 0.0804 |
| EDUC | 0.024074 | 0.006787 | 3.547204 | 0.0004 |
| EXPER | 0.011701 | 0.005183 | 2.257616 | 0.0241 |
| EXPER2 | -5.53E-05 | 6.52E-05 | -0.849150 | 0.3960 |
| EDU_EXP | -0.000478 | 0.000277 | -1.725513 | 0.0847 |
| FEMALE | -0.009757 | 0.018708 | -0.521530 | 0.6021 |
| UNION | -0.079648 | 0.025523 | -3.120623 | 0.0018 |
| NONWHITE | 0.000486 | 0.025794 | 0.018829 | 0.9850 |

# White's Test

- Estimate the OLS regression, and obtain the squared residuals
- Regress the squared residuals on all regressors, squared regressors, and pairwise regressor cross products
- To test the null hypothesis of no relationship, examine $NR^2$ from this regression. In large samples $NR^2 \sim \chi^2$ under the null.

(White's test is a natural and flexible generalization of the Breusch-Pagan-Godfrey test)

# White Test


EViews
File Edit Object View Proc Quick Options Add-ins Window Help
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Heteroskedasticity Test: White
F-statistic 2.431488 Prob. F(29,1293) 0.0000
Obs*R-squared 68.41804 Prob. Chi-Square(29) 0.0000
Scaled explained SS 91.66473 Prob. Chi-Square(29) 0.0000
-2.682949803289767
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DB

# GLS for Heteroskedasticity

- "Weighted least squares" (WLS)

  – Take a stand on the DGP. Get consistent standard errors and efficient parameter estimates.

# (Infeasible) Weighted Least Squares

DGP:

$$y_i = x_i'\beta + \varepsilon_i$$

$$\varepsilon_i \sim idN(0, \sigma_i^2)$$

Weight the data $(y_i, x_i)$ by $1/\sigma_i$:

$$\frac{y_i}{\sigma_i} = \frac{x_i'\beta}{\sigma_i} + \frac{\varepsilon_i}{\sigma_i}$$

The DGP is now:

$$y_i^* = x_i^{*\prime}\beta + \varepsilon_i^*$$

$$\varepsilon_i^* \sim iidN(0, 1)$$

- OLS is MVUE!
- Problem: We don't know $\sigma_i^2$

# Remark on Weighted Least Squares

Weighting the data by $1/\sigma_i$ is the same as weighting the residuals by $1/\sigma_i^2$:

$$\min_{\beta} \sum_{i=1}^{N} \left( \frac{y_i - x_i'\beta}{\sigma_i} \right)^2 = \min_{\beta} \sum_{i=1}^{N} \frac{1}{\sigma_i^2} \left( y_i - x_i'\beta \right)^2$$

# Feasible Weighted Least Squares

Intuition: Replace the unknown $\sigma_i^2$ values with estimates

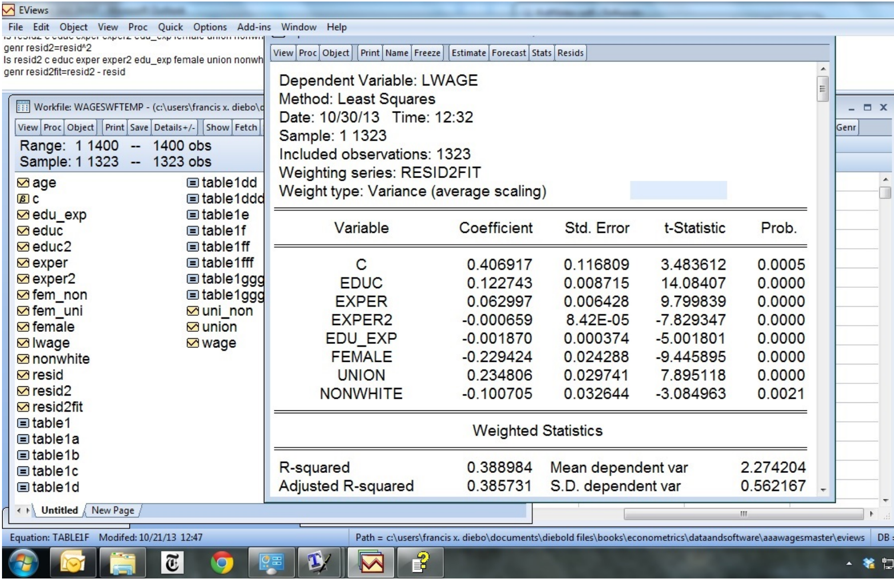
Some good ideas:

- Use $w_i = 1/\widehat{e_i^2}$, where $\widehat{e_i^2}$ are from the BGP test regression
- Use $w_i = 1/\widehat{e_i^2}$, where $\widehat{e_i^2}$ are from the White test regression

What about WLS directly using $w_i = 1/e_i^2$?

- Not such a good idea
- $e_i^2$ too noisy; we'd like to use not $e_i^2$ but rather $E(e_i^2|x_i)$. So we use an estimate of $E(e_i^2|x_i)$, namely $\widehat{e_i^2}$ from $e^2 \rightarrow X$

# Regression Weighted by Fit From White Test Regression




Dependent Variable: LWAGE
Method: Least Squares
Date: 10/30/13 Time: 12:32
Sample: 1 1323
Included observations: 1323
Weighting series: RESID2FIT
Weight type: Variance (average scaling)

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.406917 | 0.116809 | 3.483612 | 0.0005 |
| EDUC | 0.122743 | 0.008715 | 14.08407 | 0.0000 |
| EXPER | 0.062997 | 0.006428 | 9.799839 | 0.0000 |
| EXPER2 | -0.000659 | 8.42E-05 | -7.829347 | 0.0000 |
| EDU_EXP | -0.001870 | 0.000374 | -5.001801 | 0.0000 |
| FEMALE | -0.229424 | 0.024288 | -9.445895 | 0.0000 |
| UNION | 0.234806 | 0.029741 | 7.895118 | 0.0000 |
| NONWHITE | -0.100705 | 0.032644 | -3.084963 | 0.0021 |

Weighted Statistics

| | | | |
|---|---|---|---|
| R-squared | 0.388984 | Mean dependent var | 2.274204 |
| Adjusted R-squared | 0.385731 | S.D. dependent var | 0.562167 |

# A Different Approach
# (Advanced but Very Important)
# White's Heteroskedasticity-Consistent Standard Errors

Perhaps surprisingly, we make direct use of $e_i^2$

Don't take a stand on the DGP

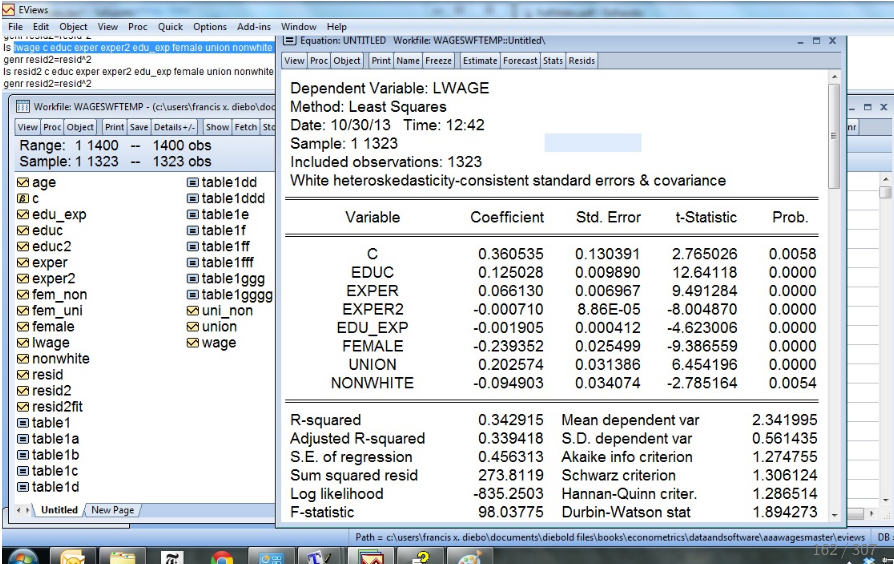
Give up on efficient parameter estimates, but get consistent s.e.'s.

Using advanced methods, one *can* obtain consistent s.e.'s (if not an efficient $\hat{\beta}$) using only $e_i^2$

- Standard errors are rendered consistent.
- $\hat{\beta}$ remains unchanged at its OLS value. (Is that a problem?)

"Robustness to heteroskedasticity of unknown form"

# Regression with White's Heteroskedasticity-Consistent Standard Errors



Dependent Variable: LWAGE
Method: Least Squares
Date: 10/30/13 Time: 12:42
Sample: 1 1323
Included observations: 1323
White heteroskedasticity-consistent standard errors & covariance

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.360535 | 0.130391 | 2.765026 | 0.0058 |
| EDUC | 0.125028 | 0.009890 | 12.64118 | 0.0000 |
| EXPER | 0.066130 | 0.006967 | 9.491284 | 0.0000 |
| EXPER2 | -0.000710 | 8.86E-05 | -8.004870 | 0.0000 |
| EDU_EXP | -0.001905 | 0.000412 | -4.623006 | 0.0000 |
| FEMALE | -0.239352 | 0.025499 | -9.386559 | 0.0000 |
| UNION | 0.202574 | 0.031386 | 6.454196 | 0.0000 |
| NONWHITE | -0.094903 | 0.034074 | -2.785164 | 0.0054 |

| | | | |
|---|---|---|---|
| R-squared | 0.342915 | Mean dependent var | 2.341995 |
| Adjusted R-squared | 0.339418 | S.D. dependent var | 0.561435 |
| S.E. of regression | 0.456313 | Akaike info criterion | 1.274755 |
| Sum squared resid | 273.8119 | Schwarz criterion | 1.306124 |
| Log likelihood | -835.2503 | Hannan-Quinn criter. | 1.286514 |
| F-statistic | 98.03775 | Durbin-Watson stat | 1.894273 |

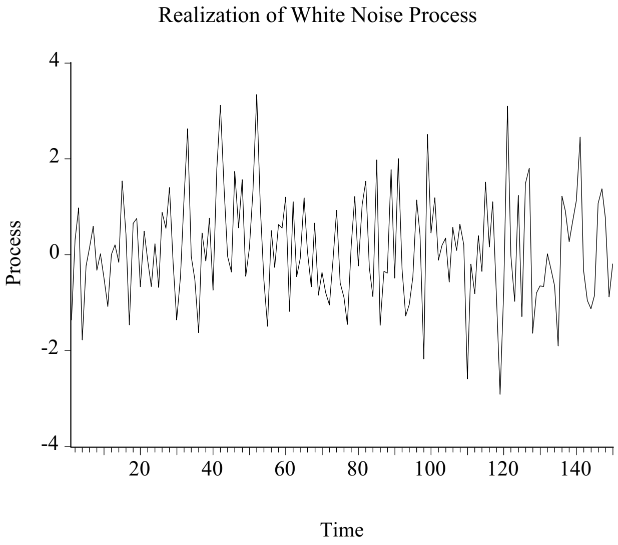
# A Tiny Bit of Time-Series Theory: White Noise and $AR(1)$ Processes

White noise: $y_t \sim WN(\mu, \sigma^2)$ (*serially uncorrelated*)

Zero-mean white noise: $y_t \sim WN(0, \ \sigma^2)$

Independent (strong) white noise: $y_t \overset{iid}{\sim} (0, \sigma^2)$

Gaussian white noise: $y_t \overset{iid}{\sim} N(0, \sigma^2)$


Realization of White Noise Process
Process
Time
4
2
0
-2
-4
20
40
60
80
100
120
140

# Autocovariance, Autocorrelation and Partial Autocorrelation Functions

Population autocovariances:

$$\gamma_y(\tau) = cov(y_t, y_{t-\tau}),\ \tau = 0, 1, 2, ...$$

Population autocorrelations:

$$\rho_y(\tau) = \frac{\gamma_y(\tau)}{\gamma_y(0)} = corr(y_t, y_{t-\tau}),\ \tau = 0, 1, 2, ...$$

Population partial autocorrelations:

$p_y(\tau)$ is the coefficient on $y_{t-\tau}$ in the projection

$$y_t \to c, y_{t-1}, ..., y_{t-(\tau-1)}, y_{t-\tau},\ \tau = 0, 1, 2, ...$$

# Moment Structure of Strong White Noise

$$E(y_t) = 0, \; var(y_t) = \sigma^2, \; E(y_t|\Omega_{t-1}) = 0$$

$$var(y_t|\Omega_{t-1}) = E[(y_t - E(y_t|\Omega_{t-1}))^2|\Omega_{t-1}] = \sigma^2$$

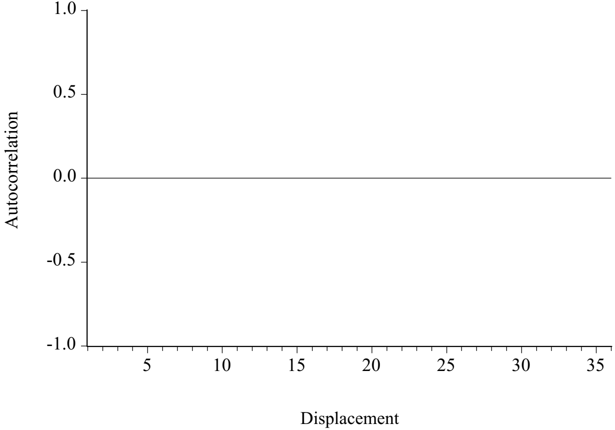
$$\text{where } \Omega_{t-1} = \{y_{t-1}, \; y_{t-2}, \; ...\}$$

$$\gamma(\tau) = \begin{cases} \sigma^2, \; \tau = 0 \\ \\ 0, \; \tau \geq 1 \end{cases}$$

$$\rho(\tau) = \begin{cases} 1, \; \tau = 0 \\ \\ 0, \; \tau \geq 1 \end{cases}$$

$$p(\tau) = \begin{cases} 1, \; \tau = 0 \\ \\ 0, \; \tau \geq 1 \end{cases}$$

Population Autocorrelation Function
White Noise Process



# Zero-Mean $AR(1)$

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim iidN(0,\ \sigma^2),\ \ |\phi| < 1$$

– Regression on just a lagged dependent variable

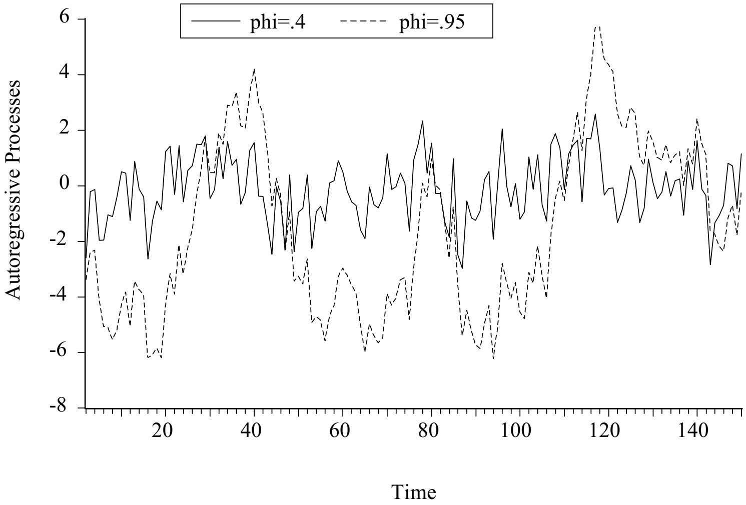
– "Autoregression"

Back-substitution reveals that:

$$y_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$$

$$\implies\ E(y_t) = 0$$

# Realizations of Zero-Mean Two $AR(1)$ Processes



# Moment Structure of the Zero-Mean $AR(1)$ Process

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$E(y_t) = 0 \text{ (of course)}$$

$$var(y_t) = \frac{\sigma^2}{1-\phi^2} \text{ (hmmm...)}$$

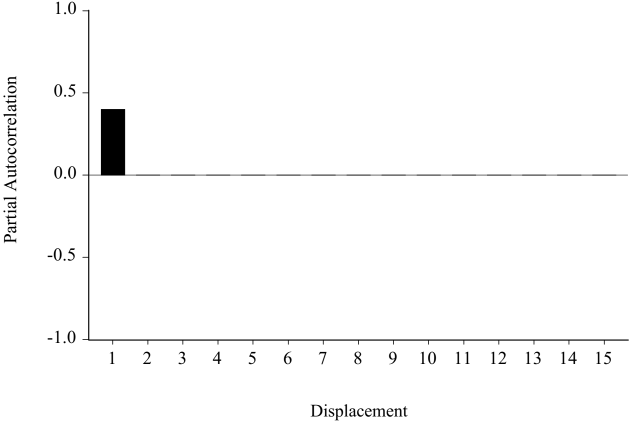
$$E(y_t|\Omega_{t-1}) = \phi y_{t-1} \text{ (obvious)}$$

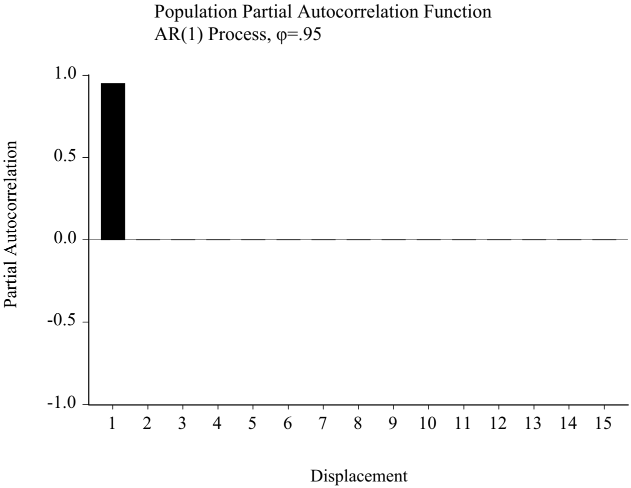
$$var(y_t|\Omega_{t-1}) = \sigma^2 \text{ (obvious)}$$

$$\rho(\tau) = \begin{cases} 1, \ \tau = 0 \\ \phi^\tau, \ \tau \geq 1 \end{cases} \quad \text{(hmmm...)}$$

$$p(\tau) = \begin{cases} 1, \ \tau = 0 \\ \phi, \ \tau = 1 \\ 0, \ \tau \geq 2 \end{cases} \quad \text{(obvious)}$$

Population Partial Autocorrelation Function
AR(1) Process, φ=.4




Population Partial Autocorrelation Function
AR(1) Process, φ=.95
Partial Autocorrelation
1.0
0.5
0.0
-0.5
-1.0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
Displacement

# $AR(1)$ Autocorrelation Function

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$\Longrightarrow \quad y_t y_{t-\tau} = \phi y_{t-1} y_{t-\tau} + \varepsilon_t y_{t-\tau} \quad (1)$$

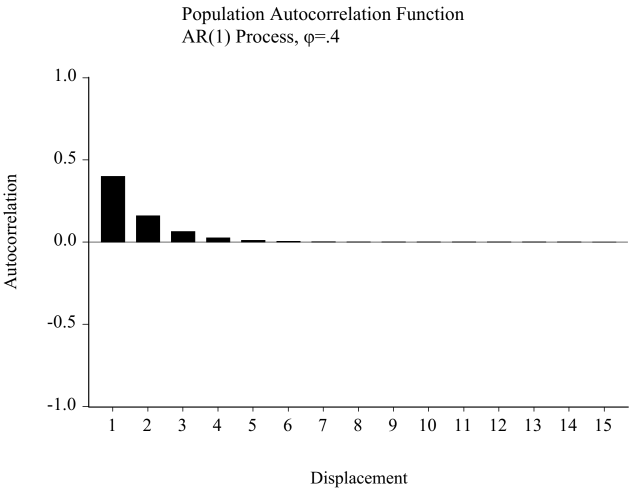
First consider $\tau = 0$. Immediately:

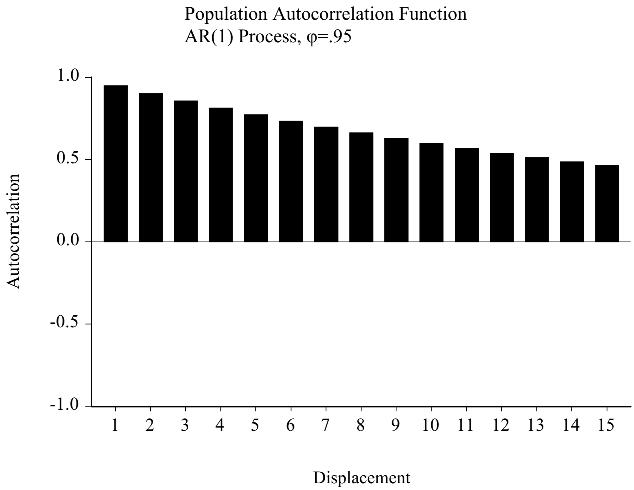
$$\gamma(0) = var(y_t) = \frac{\sigma^2}{1-\phi^2}$$

Now consider $\tau > 0$. Taking expectations of (1) produces:

$$\gamma(\tau) = \phi\gamma(\tau - 1)$$

Hence $\gamma(\tau) = \phi^\tau \frac{\sigma^2}{1-\phi^2}$, so $\rho(\tau) = \phi^\tau, \ \tau = 0, 1, 2, ...$


Population Autocorrelation Function
AR(1) Process, φ=.4
Autocorrelation
1.0
0.5
0.0
-0.5
-1.0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Displacement


Population Autocorrelation Function
AR(1) Process, φ=.95
1.0
0.5
0.0
-0.5
-1.0
Autocorrelation
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
Displacement

# $\gamma(\tau)$, $\rho(\tau)$, and $p(\tau)$ for Generic $AR(p)$

$$AR(p) Process:$$

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_p y_{t-p} + \varepsilon_t$$

$$\gamma(\tau) \rightarrow 0 \text{ as } \tau \rightarrow \infty, \textit{ gradually}$$

$$\rho(\tau) \rightarrow 0 \text{ as } \tau \rightarrow \infty, \textit{ gradually}$$

$$p(\tau) \rightarrow 0 \text{ at } \tau = p, \textit{ sharply}$$

# Non-Zero Mean I ($AR(1)$ Example): Regression on an Intercept and $y_{t-1}$, With White Noise Disturbances

$$(y_t - \mu) = \phi(y_{t-1} - \mu) + \varepsilon_t$$

$$\varepsilon_t \sim iidN(0,\ \sigma^2),\ \ |\phi| < 1$$

$$\implies y_t = c + \phi y_{t-1} + \varepsilon_t, \text{ where } c = \mu(1 - \phi)$$

Back-substitution reveals that:

$$y_t = \mu + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j}$$

$$\implies E(y_t) = \mu$$

# Non-Zero Mean II ($AR(1)$ Example, Cont'd): Regression on an Intercept Alone, with AR(1) Disturbances

$$y_t = \mu + \varepsilon_t$$

$$\varepsilon_t = \phi \varepsilon_{t-1} + v_t$$

$$v_t \sim iidN(0, \sigma^2), \;\; |\phi| < 1$$

# The Sample Autocorrelation Function

Autocorrelations:

$$\rho_y(\tau) = corr(y_t, y_{t-\tau}) = \frac{cov(y_t, y_{t-\tau})}{\sqrt{var(y_t)}\sqrt{var(y_{t-\tau})}} = \frac{cov(y_t, y_{t-\tau})}{var(y_t)}$$

Sample autocorrelations:

$$\hat{\rho}_y(\tau) = \frac{\widehat{cov}(y_t, y_{t-\tau})}{\widehat{var}(y_t)} = \frac{\frac{1}{T}\sum_t y_t y_{t-\tau}}{\frac{1}{T}\sum_t y_t^2}$$

We view $\hat{\rho}_y(\tau)$ as a function of $\tau$ and examine its shape.

# The Sample Partial Autocorrelation Function

Partial autocorrelations:

$\hat{p}_y(\tau)$ is the coefficient on $y_{t-\tau}$ in the projection

$$y_t \rightarrow c, y_{t-1}, ..., y_{t-(\tau-1)}, y_{t-\tau},\ \tau = 0, 1, 2, ...$$

Sample partial autocorrelations:

$\hat{p}_y(\tau)$ is the coefficient on $y_{t-\tau}$ in the regression

$$y_t \rightarrow c, y_{t-1}, ..., y_{t-(\tau-1)}, y_{t-\tau},\ t = 1, ..., T,\ \tau = 0, 1, 2, ...$$

We view $\hat{\rho}_y(\tau)$ as a function of $\tau$ and examine its shape.

# Bartlett Standard Errors

Under $H_0: \; y_t \sim iidN(0, \sigma^2)$, we have (as $T \to \infty$):

$$(1) \; \hat{\rho}_y(\tau) \overset{a}{\sim} N\left(0, \frac{1}{T}\right), \; \forall \tau$$

(used for inference on individual autocorrelations)
95% "Bartlett bands" under the *iid* null: $0 \pm \frac{2}{\sqrt{T}}$

$$(2) \; cov(\hat{\rho}_y(\tau), \hat{\rho}_y(\tau + v)) = 0, \; \forall \tau, \; v$$

(used to derive distributions of Box-Pierce and Ljung-Box stats)

# Box-Pierce and Ljung-Box $Q$ Statistics

Under $H_0: \; y_t \sim iidN(0, \sigma^2)$, we have (as $T \to \infty$):

$$Q_{BP} = T \sum_{\tau=1}^{m} \hat{\rho}^2(\tau) \sim \chi^2_m$$

$$Q_{LB} = T(T+2) \sum_{\tau=1}^{m} \left( \frac{1}{T - \tau} \right) \hat{\rho}^2(\tau) \sim \chi^2_m$$

(We test an *implication* of *iid*, $\rho(1) = \rho(2) = \ldots = \rho(m) = 0$)

# (Part of a) Correlogram

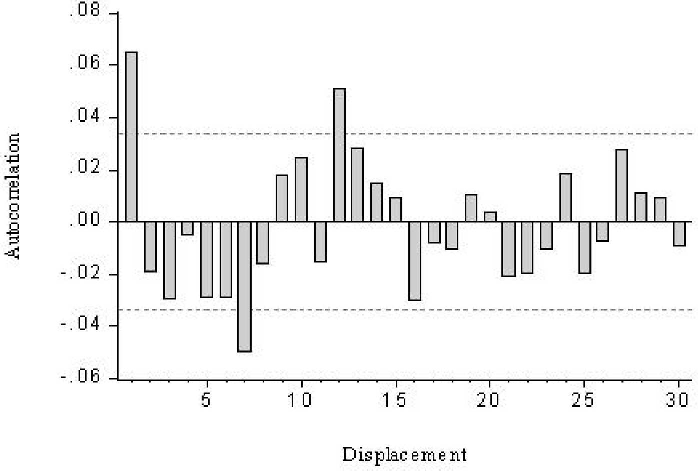



Figure: Sample Acorr Fn, Daily Stock Market Returns

# Serial Correlation in Time-Series Regression

Consider:

$$\varepsilon \sim N(\underline{0}, \sigma^2\Omega)$$

The FIC case is $\Omega = I$. When is $\Omega \neq I$?

We've already seen heteroskedasticity.

Now we consider "serial correlation" or "autocorrelation."

$\rightarrow \quad \varepsilon_t$ is correlated with $\varepsilon_{t-\tau} \quad \leftarrow$

Can arise for many reasons, but they all boil down to:

The included $X$ variables fail to capture all the dynamics in $y$.

– No additional explanation needed!

# On $\Omega$ with Heteroskedasticity vs. Serial Correlation

With heteroskedasticity, $\varepsilon_i$ is independent across $i$ but not identically distributed across $i$ (variance of $\varepsilon_i$ varies with $i$):

$$\sigma^2\Omega = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N^2 \end{pmatrix}$$

With serial correlation, $\varepsilon_t$ is correlated across $t$
but unconditionally identically distributed across $t$:

$$\sigma^2\Omega = \begin{pmatrix} \sigma^2 & \gamma(1) & \dots & \gamma(T-1) \\ \gamma(1) & \sigma^2 & \dots & \gamma(T-2) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(T-1) & \gamma(T-2) & \dots & \sigma^2 \end{pmatrix}$$

# Consequences of Serial Correlation

OLS inefficient (no longer BLUE),
in finite samples and asymptotically

Standard errors biased and inconsistent. Hence $t$ ratios do not have the $t$ distribution in finite samples and do not have the $N(0, 1)$ distribution asymptotically

Does this sound familiar?

# Detection

- Graphical autocorrelation diagnostics
  - Residual plot
  - Scatterplot of $e_t$ against $e_{t-\tau}$

- Formal autocorrelation tests and analyses
  - Durbin-Watson
  - Breusch-Godfrey
  - Residual correlogram

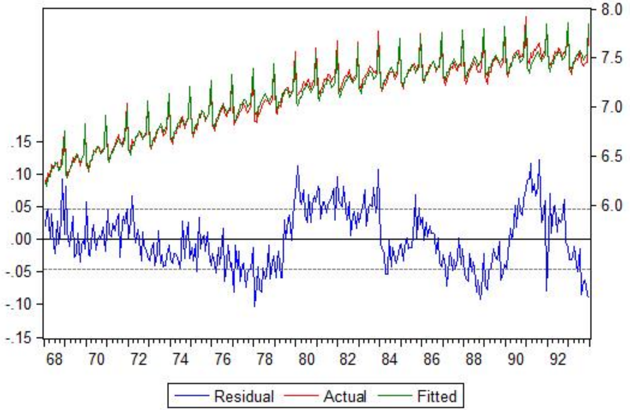
# Liquor Sales Regression on Trend and Seasonals

Dependent Variable: LSALES
Method: Least Squares
Date: 10/13/12 Time: 12:32
Sample: 1968M01 1993M12
Included observations: 312

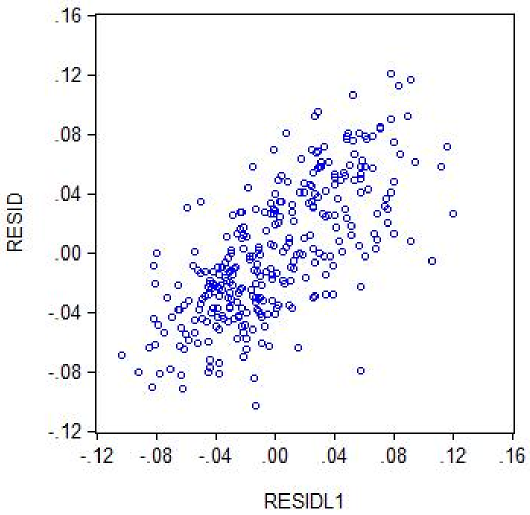
| Variable | Coef | Std. Err | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 0.007656 | 0.000123 | 62.35882 | 0.0000 |
| TIME2 | -1.14E-05 | 3.56E-07 | -32.06823 | 0.0000 |
| D1 | 6.147456 | 0.012340 | 498.1699 | 0.0000 |
| D2 | 6.088653 | 0.012353 | 492.8890 | 0.0000 |
| D3 | 6.174127 | 0.012366 | 499.3008 | 0.0000 |
| D4 | 6.175220 | 0.012378 | 498.8970 | 0.0000 |
| D5 | 6.246086 | 0.012390 | 504.1398 | 0.0000 |
| D6 | 6.250387 | 0.012401 | 504.0194 | 0.0000 |
| D7 | 6.295979 | 0.012412 | 507.2402 | 0.0000 |
| D8 | 6.268043 | 0.012423 | 504.5509 | 0.0000 |
| D9 | 6.203832 | 0.012433 | 498.9630 | 0.0000 |
| D10 | 6.229197 | 0.012444 | 500.5968 | 0.0000 |
| D11 | 6.259770 | 0.012453 | 502.6602 | 0.0000 |
| D12 | 6.580068 | 0.012463 | 527.9819 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.986111 | Mean dependent var | 7.112383 |
| Adjusted R-squared | 0.985505 | S.D. dependent var | 0.379308 |
| S.E. of regression | 0.045666 | Akaike info criterion | -3.291086 |
| Sum squared resid | 0.621448 | Schwarz criterion | -3.123131 |
| Log likelihood | 527.4094 | Hannan-Quinn criter. | -3.223959 |
| Durbin-Watson stat | 0.586187 | | |

# Graphical Diagnostics - Residual Plot



# Graphical Diagnostics - Scatterplot of $e_t$ against $e_{t-1}$



# Formal Tests and Analyses: Durbin-Watson (0.59!)

Simple paradigm ($AR(1)$):

$$y_t = x_t'\beta + \varepsilon_t$$

$$\varepsilon_t = \phi\varepsilon_{t-1} + v_t$$

$$v_t \sim iid\ N(0,\ \sigma^2)$$

We want to test $H_0:\ \phi = 0$ against $H_1:\ \phi \neq 0$

Regress $y \rightarrow X$ and obtain the residuals $e_t$

$$DW = \frac{\sum_{t=2}^{T}(e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2}$$

# Understanding the Durbin-Watson Statistic

$$DW = \frac{\sum_{t=2}^{T}(e_t - e_{t-1})^2}{\sum_{t=1}^{T} e_t^2} = \frac{\frac{1}{T}\sum_{t=2}^{T}(e_t - e_{t-1})^2}{\frac{1}{T}\sum_{t=1}^{T} e_t^2}$$

$$= \frac{\frac{1}{T}\sum_{t=2}^{T} e_t^2 + \frac{1}{T}\sum_{t=2}^{T} e_{t-1}^2 - 2\ \frac{1}{T}\sum_{t=2}^{T} e_t e_{t-1}}{\frac{1}{T}\sum_{t=1}^{T} e_t^2}$$

Hence as $T \rightarrow \infty$:

$$DW \approx \frac{\sigma^2 + \sigma^2 - 2cov(e_t, e_{t-1})}{\sigma^2} = 2(1 - \underbrace{corr(e_t, e_{t-1})}_{\rho_e(1)})$$

$$\implies DW \in [0, 4],\ DW \rightarrow 2 \text{ as } \phi \rightarrow 0, \text{ and } DW \rightarrow 0 \text{ as } \phi \rightarrow 1$$

# Formal Tests and Analyses: Breusch-Godfrey

General $AR(p)$ environment:

$$y_t = x_t'\beta + \varepsilon_t$$

$$\varepsilon_t = \phi_1\varepsilon_{t-1} + ... + \phi_p\varepsilon_{t-p} + v_t$$

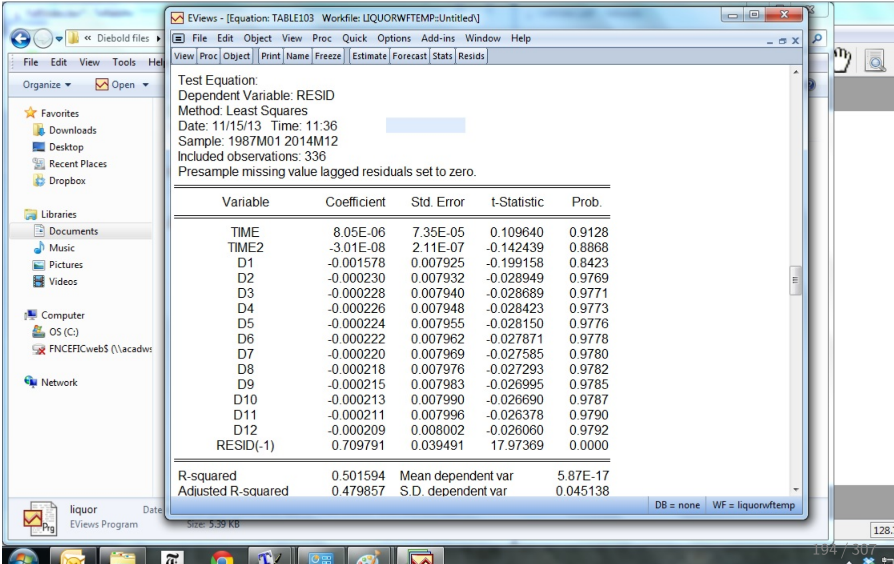
$$v_t \sim iidN(0, \sigma^2)$$

We want to test $H_0: \ (\phi_1, ..., \phi_p) = \underline{0}$ against $H_1: \ (\phi_1, ..., \phi_p) \neq \underline{0}$

- Regress $y_t \rightarrow x_t$ and obtain the residuals $e_t$
- Regress $e_t \rightarrow x_t, e_{t-1}, ..., e_{t-p}$
- Examine $TR^2$. In large samples $TR^2 \sim \chi^2_p$ under the null.

Does this sound familiar?

# BG for $AR(1)$ Disturbances
# ($TR^2 = 168.5$, $p = 0.0000$)



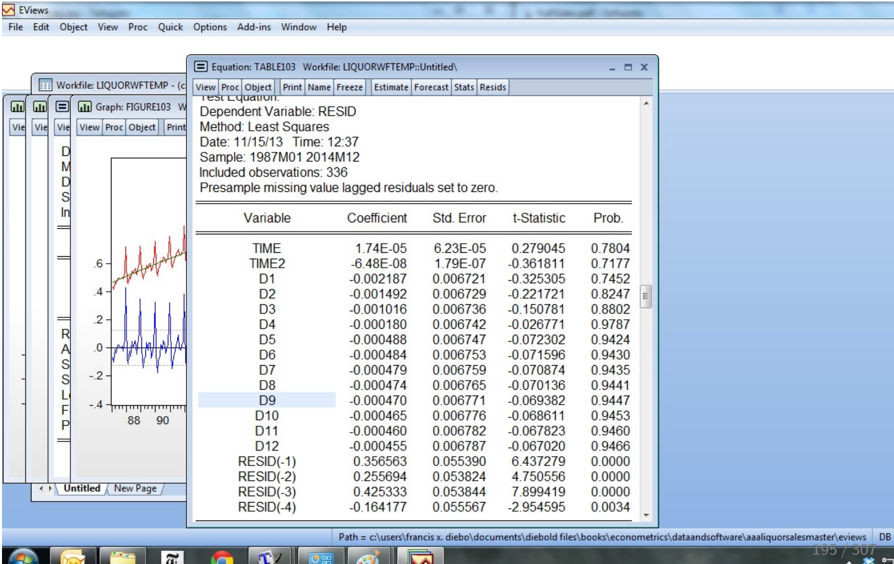
EViews - [Equation: TABLE103 Workfile: LIQUORWFTEMP::Untitled\]

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 11/15/13 Time: 11:36
Sample: 1987M01 2014M12
Included observations: 336
Presample missing value lagged residuals set to zero.

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 8.05E-06 | 7.35E-05 | 0.109640 | 0.9128 |
| TIME2 | -3.01E-08 | 2.11E-07 | -0.142439 | 0.8868 |
| D1 | -0.001578 | 0.007925 | -0.199158 | 0.8423 |
| D2 | -0.000230 | 0.007932 | -0.028949 | 0.9769 |
| D3 | -0.000228 | 0.007940 | -0.028689 | 0.9771 |
| D4 | -0.000226 | 0.007948 | -0.028423 | 0.9773 |
| D5 | -0.000224 | 0.007955 | -0.028150 | 0.9776 |
| D6 | -0.000222 | 0.007962 | -0.027871 | 0.9778 |
| D7 | -0.000220 | 0.007969 | -0.027585 | 0.9780 |
| D8 | -0.000218 | 0.007976 | -0.027293 | 0.9782 |
| D9 | -0.000215 | 0.007983 | -0.026995 | 0.9785 |
| D10 | -0.000213 | 0.007990 | -0.026690 | 0.9787 |
| D11 | -0.000211 | 0.007996 | -0.026378 | 0.9790 |
| D12 | -0.000209 | 0.008002 | -0.026060 | 0.9792 |
| RESID(-1) | 0.709791 | 0.039491 | 17.97369 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.501594 | Mean dependent var | 5.87E-17 |
| Adjusted R-squared | 0.479857 | S.D. dependent var | 0.045138 |

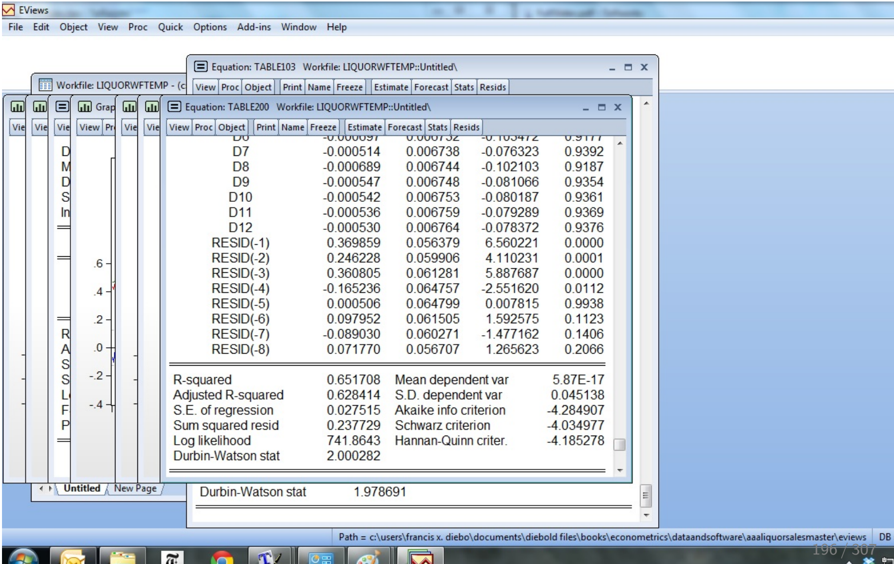
# BG for $AR(4)$ Disturbances ($TR^2 = 216.7$, $p = 0.0000$)




Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 11/15/13 Time: 12:37
Sample: 1987M01 2014M12
Included observations: 336
Presample missing value lagged residuals set to zero.

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 1.74E-05 | 6.23E-05 | 0.279045 | 0.7804 |
| TIME2 | -6.48E-08 | 1.79E-07 | -0.361811 | 0.7177 |
| D1 | -0.002187 | 0.006721 | -0.325305 | 0.7452 |
| D2 | -0.001492 | 0.006729 | -0.221721 | 0.8247 |
| D3 | -0.001016 | 0.006736 | -0.150781 | 0.8802 |
| D4 | -0.000180 | 0.006742 | -0.026771 | 0.9787 |
| D5 | -0.000488 | 0.006747 | -0.072302 | 0.9424 |
| D6 | -0.000484 | 0.006753 | -0.071596 | 0.9430 |
| D7 | -0.000479 | 0.006759 | -0.070874 | 0.9435 |
| D8 | -0.000474 | 0.006765 | -0.070136 | 0.9441 |
| D9 | -0.000470 | 0.006771 | -0.069382 | 0.9447 |
| D10 | -0.000465 | 0.006776 | -0.068611 | 0.9453 |
| D11 | -0.000460 | 0.006782 | -0.067823 | 0.9460 |
| D12 | -0.000455 | 0.006787 | -0.067020 | 0.9466 |
| RESID(-1) | 0.356563 | 0.055390 | 6.437279 | 0.0000 |
| RESID(-2) | 0.255694 | 0.053824 | 4.750556 | 0.0000 |
| RESID(-3) | 0.425333 | 0.053844 | 7.899419 | 0.0000 |
| RESID(-4) | -0.164177 | 0.055567 | -2.954595 | 0.0034 |

# BG for $AR(8)$ Disturbances
## $(TR^2 = 219.0,\ p = 0.0000)$




| | | | | |
|---|---|---|---|---|
| D7 | -0.000514 | 0.006738 | -0.076323 | 0.9392 |
| D8 | -0.000689 | 0.006744 | -0.102103 | 0.9187 |
| D9 | -0.000547 | 0.006748 | -0.081066 | 0.9354 |
| D10 | -0.000542 | 0.006753 | -0.080187 | 0.9361 |
| D11 | -0.000536 | 0.006759 | -0.079289 | 0.9369 |
| D12 | -0.000530 | 0.006764 | -0.078372 | 0.9376 |
| RESID(-1) | 0.369859 | 0.056379 | 6.560221 | 0.0000 |
| RESID(-2) | 0.246228 | 0.059906 | 4.110231 | 0.0001 |
| RESID(-3) | 0.360805 | 0.061281 | 5.887687 | 0.0000 |
| RESID(-4) | -0.165236 | 0.064757 | -2.551620 | 0.0112 |
| RESID(-5) | 0.000506 | 0.064799 | 0.007815 | 0.9938 |
| RESID(-6) | 0.097952 | 0.061505 | 1.592575 | 0.1123 |
| RESID(-7) | -0.089030 | 0.060271 | -1.477162 | 0.1406 |
| RESID(-8) | 0.071770 | 0.056707 | 1.265623 | 0.2066 |

| | | | |
|---|---|---|---|
| R-squared | 0.651708 | Mean dependent var | 5.87E-17 |
| Adjusted R-squared | 0.628414 | S.D. dependent var | 0.045138 |
| S.E. of regression | 0.027515 | Akaike info criterion | -4.284907 |
| Sum squared resid | 0.237729 | Schwarz criterion | -4.034977 |
| Log likelihood | 741.8643 | Hannan-Quinn criter. | -4.185278 |
| Durbin-Watson stat | 2.000282 | | |

Durbin-Watson stat 1.978691

# Formal Tests and Analyses: Residual Correlogram

$$\hat{\rho}_e(\tau) = \frac{\widehat{cov}(e_t, e_{t-\tau})}{\widehat{var}(e_t)} = \frac{\frac{1}{T}\sum_t e_t e_{t-\tau}}{\frac{1}{T}\sum_t e_t^2}$$

$\hat{p}_e(\tau)$ is the coefficient on $e_{t-\tau}$ in the regression

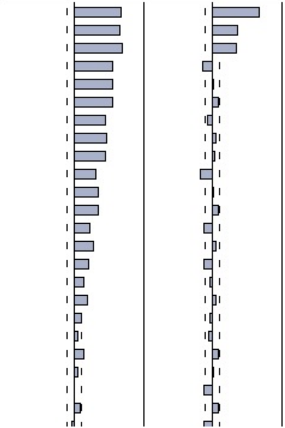
$$e_t \rightarrow c, e_{t-1}, ..., e_{t-(\tau-1)}, e_{t-\tau}$$

Approximate 95% "Bartlett bands" under the *iid N* null: $0 \pm \frac{2}{\sqrt{T}}$

$$Q_{BP} = T \sum_{\tau=1}^{m} \hat{\rho}_e^2(\tau) \sim \chi^2_{m-K} \text{ under } iid\ N$$

$$Q_{LB} = T(T+2) \sum_{\tau=1}^{m} \left( \frac{1}{T-\tau} \right) \hat{\rho}_e^2(\tau) \sim \chi^2_{m-K}$$

# Residual Correlogram for Trend + Seasonal Model



Date: 10/14/12 Time: 18:32
Sample: 1968M01 1993M12
Included observations: 312

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | 0.700 | 0.700 | 154.34 | 0.000 |
| | | 2 | 0.686 | 0.383 | 302.86 | 0.000 |
| | | 3 | 0.725 | 0.369 | 469.36 | 0.000 |
| | | 4 | 0.569 | -0.141 | 572.36 | 0.000 |
| | | 5 | 0.569 | 0.017 | 675.58 | 0.000 |
| | | 6 | 0.577 | 0.093 | 782.19 | 0.000 |
| | | 7 | 0.460 | -0.078 | 850.06 | 0.000 |
| | | 8 | 0.480 | 0.043 | 924.38 | 0.000 |
| | | 9 | 0.466 | 0.030 | 994.46 | 0.000 |
| | | 10 | 0.327 | -0.188 | 1029.1 | 0.000 |
| | | 11 | 0.364 | 0.019 | 1072.1 | 0.000 |
| | | 12 | 0.355 | 0.089 | 1113.3 | 0.000 |
| | | 13 | 0.225 | -0.119 | 1129.9 | 0.000 |
| | | 14 | 0.291 | 0.065 | 1157.8 | 0.000 |
| | | 15 | 0.211 | -0.119 | 1172.4 | 0.000 |
| | | 16 | 0.138 | -0.031 | 1178.7 | 0.000 |
| | | 17 | 0.195 | 0.053 | 1191.4 | 0.000 |
| | | 18 | 0.114 | -0.027 | 1195.7 | 0.000 |
| | | 19 | 0.055 | -0.063 | 1196.7 | 0.000 |
| | | 20 | 0.134 | 0.089 | 1202.7 | 0.000 |
| | | 21 | 0.062 | 0.018 | 1204.0 | 0.000 |
| | | 22 | -0.006 | -0.115 | 1204.0 | 0.000 |
| | | 23 | 0.084 | 0.086 | 1206.4 | 0.000 |
| | | 24 | [illegible] | [illegible] | [illegible] | [illegible] |

# Correcting for Autocorrelation

- Generalized least squares
  – Transform the data such that the classical conditions hold

- Heteroskedasticity and autocorrelation consistent (HAC) s.e.'s
  – Use OLS, but calculate standard errors robustly

# Recall Generalized Least Squares (*GLS*)

Consider the FIC except that we now let:

$$\varepsilon \sim N(\underline{0}, \sigma^2\Omega)$$

The GLS estimator is:

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$$

Under the remaining full ideal conditions it is consistent, normally distributed with covariance matrix $\sigma^2(X'\Omega^{-1}X)^{-1}$, and MVUE:

$$\hat{\beta}_{GLS} \sim N\left(\beta,\ \sigma^2(X'\Omega^{-1}X)^{-1}\right)$$

# Infeasible *GLS* (Illustrated in the Durbin-Watson $AR(1)$ Environment)

$$y_t = x_t'\beta + \varepsilon_t \quad (1a)$$

$$\varepsilon_t = \phi\varepsilon_{t-1} + v_t \quad (1b)$$

$$v_t \sim iid\, N(0, \sigma^2) \quad (1c)$$

Suppose that you know $\phi$. Then you could form:

$$\phi y_{t-1} = \phi x_{t-1}'\beta + \phi\varepsilon_{t-1} \quad (1a*)$$

$$\implies (y_t - \phi y_{t-1}) = (x_t' - \phi x_{t-1}')\beta + (\varepsilon_t - \phi\varepsilon_{t-1}) \text{ (just (1a) – (1a}*\text{))}$$

$$\implies y_t = \phi y_{t-1} + x_t'\beta - x_{t-1}'(\phi\beta) + v_t$$

– Satisfies the classical conditions! Note the restriction.

So, two key closely-related regressions:

$$y_t \to x_t \text{ (with } AR(1) \text{ disturbances)}$$

$$y_t \to y_{t-1}, x_t, x_{t-1} \text{ (with } WN \text{ disturbances and a coef. restr.)}$$

# Feasible *GLS*

(1) Replace the unknown $\phi$ value with an estimate and run the OLS regression:

$$(y_t - \hat{\phi} y_{t-1}) \rightarrow (x_t' - \hat{\phi} x_{t-1}')$$

– Iterate if desired: $\hat{\beta}_1, \hat{\phi}_1, \hat{\beta}_2, \hat{\phi}_2, ...$

(2) Run the OLS Regression

$$y_t \rightarrow y_{t-1}, x_t, x_{t-1}$$

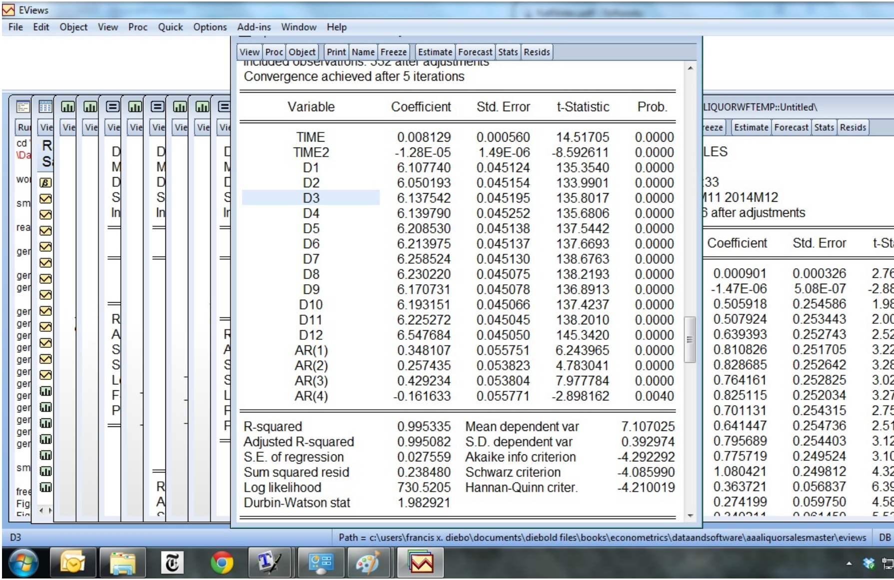
subject to the constraint noted earlier (or not)

– Generalizes trivially to $AR(p)$:

$$y_t \rightarrow y_{t-1}, ..., y_{t-p}, x_t, x_{t-1}, ..., x_{t-p}$$

(Select $p$ using the usual *AIC*, *SIC*, etc.)

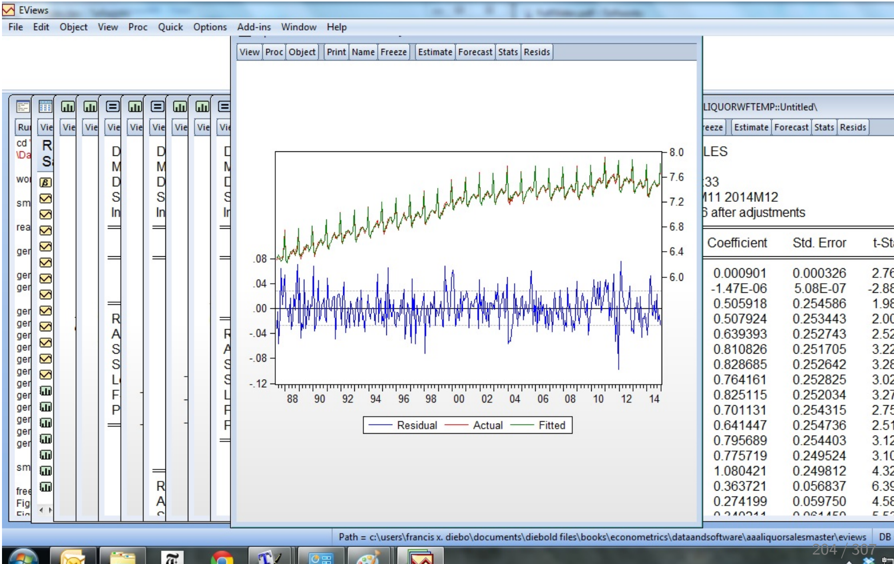
# Trend + Seasonal Model with $AR(4)$ Disturbances



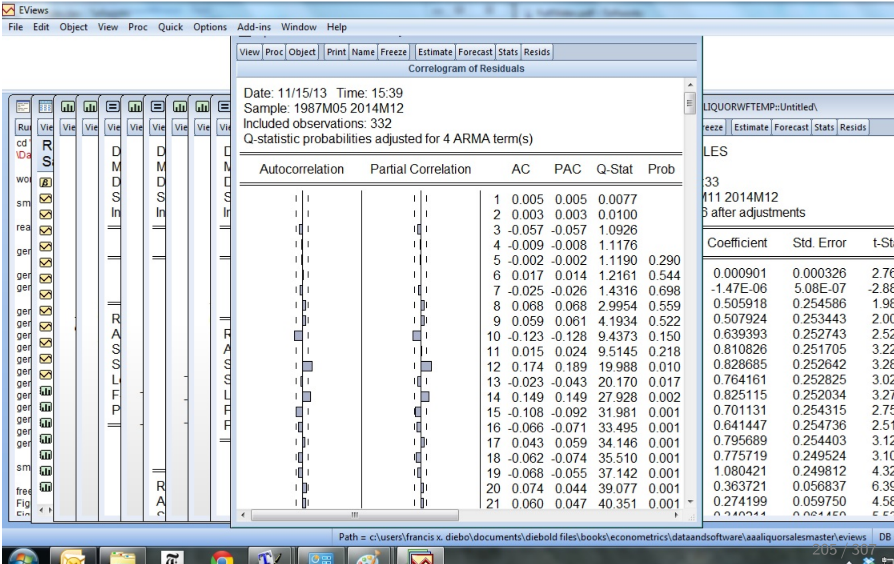

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 0.008129 | 0.000560 | 14.51705 | 0.0000 |
| TIME2 | -1.28E-05 | 1.49E-06 | -8.592611 | 0.0000 |
| D1 | 6.107740 | 0.045124 | 135.3540 | 0.0000 |
| D2 | 6.050193 | 0.045154 | 133.9901 | 0.0000 |
| D3 | 6.137542 | 0.045195 | 135.8017 | 0.0000 |
| D4 | 6.139790 | 0.045252 | 135.6806 | 0.0000 |
| D5 | 6.208530 | 0.045138 | 137.5442 | 0.0000 |
| D6 | 6.213975 | 0.045137 | 137.6693 | 0.0000 |
| D7 | 6.258524 | 0.045130 | 138.6763 | 0.0000 |
| D8 | 6.230220 | 0.045075 | 138.2193 | 0.0000 |
| D9 | 6.170731 | 0.045078 | 136.8913 | 0.0000 |
| D10 | 6.193151 | 0.045066 | 137.4237 | 0.0000 |
| D11 | 6.225272 | 0.045045 | 138.2010 | 0.0000 |
| D12 | 6.547684 | 0.045050 | 145.3420 | 0.0000 |
| AR(1) | 0.348107 | 0.055751 | 6.243965 | 0.0000 |
| AR(2) | 0.257435 | 0.053823 | 4.783041 | 0.0000 |
| AR(3) | 0.429234 | 0.053804 | 7.977784 | 0.0000 |
| AR(4) | -0.161633 | 0.055771 | -2.898162 | 0.0040 |

| | | | |
|---|---|---|---|
| R-squared | 0.995335 | Mean dependent var | 7.107025 |
| Adjusted R-squared | 0.995082 | S.D. dependent var | 0.392974 |
| S.E. of regression | 0.027559 | Akaike info criterion | -4.292292 |
| Sum squared resid | 0.238480 | Schwarz criterion | -4.085990 |
| Log likelihood | 730.5205 | Hannan-Quinn criter. | -4.210019 |
| Durbin-Watson stat | 1.982921 | | |

# Trend + Seasonal Model with *AR*(4) Disturbances
# Residual Plot

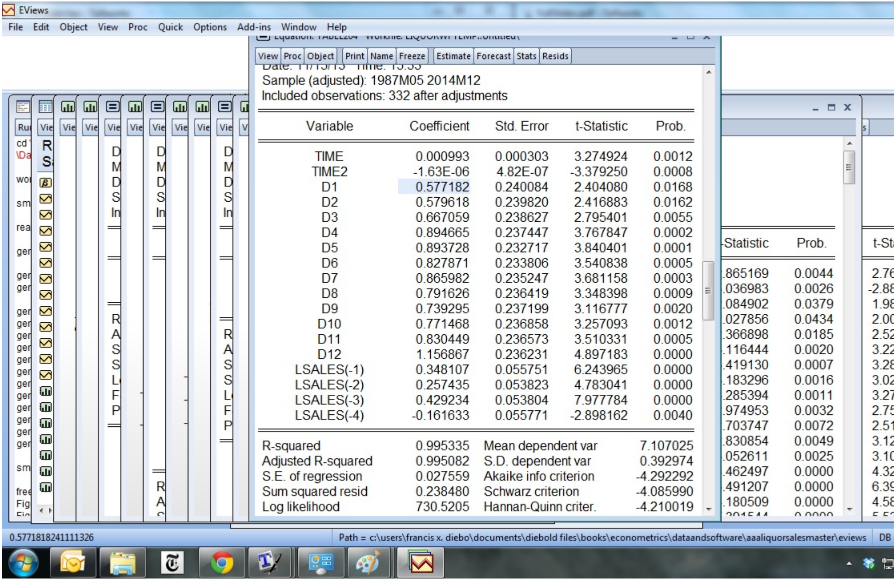


# Trend + Seasonal Model with $AR(4)$ Disturbances
# Residual Correlogram




| | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|
| 1 | 0.005 | 0.005 | 0.0077 | |
| 2 | 0.003 | 0.003 | 0.0100 | |
| 3 | -0.057 | -0.057 | 1.0926 | |
| 4 | -0.009 | -0.008 | 1.1176 | |
| 5 | -0.002 | -0.002 | 1.1190 | 0.290 |
| 6 | 0.017 | 0.014 | 1.2161 | 0.544 |
| 7 | -0.025 | -0.026 | 1.4316 | 0.698 |
| 8 | 0.068 | 0.068 | 2.9954 | 0.559 |
| 9 | 0.059 | 0.061 | 4.1934 | 0.522 |
| 10 | -0.123 | -0.128 | 9.4373 | 0.150 |
| 11 | 0.015 | 0.024 | 9.5145 | 0.218 |
| 12 | 0.174 | 0.189 | 19.988 | 0.010 |
| 13 | -0.023 | -0.043 | 20.170 | 0.017 |
| 14 | 0.149 | 0.149 | 27.928 | 0.002 |
| 15 | -0.108 | -0.092 | 31.981 | 0.001 |
| 16 | -0.066 | -0.071 | 33.495 | 0.001 |
| 17 | 0.043 | 0.059 | 34.146 | 0.001 |
| 18 | -0.062 | -0.074 | 35.510 | 0.001 |
| 19 | -0.068 | -0.055 | 37.142 | 0.001 |
| 20 | 0.074 | 0.044 | 39.077 | 0.001 |
| 21 | 0.060 | 0.047 | 40.351 | 0.001 |

# Trend + Seasonal Model with Four Lags of Dep. Var.



Sample (adjusted): 1987M05 2014M12
Included observations: 332 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 0.000993 | 0.000303 | 3.274924 | 0.0012 |
| TIME2 | -1.63E-06 | 4.82E-07 | -3.379250 | 0.0008 |
| D1 | 0.577182 | 0.240084 | 2.404080 | 0.0168 |
| D2 | 0.579618 | 0.239820 | 2.416883 | 0.0162 |
| D3 | 0.667059 | 0.238627 | 2.795401 | 0.0055 |
| D4 | 0.894665 | 0.237447 | 3.767847 | 0.0002 |
| D5 | 0.893728 | 0.232717 | 3.840401 | 0.0001 |
| D6 | 0.827871 | 0.233806 | 3.540838 | 0.0005 |
| D7 | 0.865982 | 0.235247 | 3.681158 | 0.0003 |
| D8 | 0.791626 | 0.236419 | 3.348398 | 0.0009 |
| D9 | 0.739295 | 0.237199 | 3.116777 | 0.0020 |
| D10 | 0.771468 | 0.236858 | 3.257093 | 0.0012 |
| D11 | 0.830449 | 0.236573 | 3.510331 | 0.0005 |
| D12 | 1.156867 | 0.236231 | 4.897183 | 0.0000 |
| LSALES(-1) | 0.348107 | 0.055751 | 6.243965 | 0.0000 |
| LSALES(-2) | 0.257435 | 0.053823 | 4.783041 | 0.0000 |
| LSALES(-3) | 0.429234 | 0.053804 | 7.977784 | 0.0000 |
| LSALES(-4) | -0.161633 | 0.055771 | -2.898162 | 0.0040 |

| | | | |
|---|---|---|---|
| R-squared | 0.995335 | Mean dependent var | 7.107025 |
| Adjusted R-squared | 0.995082 | S.D. dependent var | 0.392974 |
| S.E. of regression | 0.027559 | Akaike info criterion | -4.292292 |
| Sum squared resid | 0.238480 | Schwarz criterion | -4.085990 |
| Log likelihood | 730.5205 | Hannan-Quinn criter. | -4.210019 |

# How Did we Arrive at $AR(4)$ Dynamics?

Everything points there:

– Supported by original trend + seasonal residual correlogram

– Supported by $DW$

– Supported by $BG$

– Supported by $SIC$ pattern:

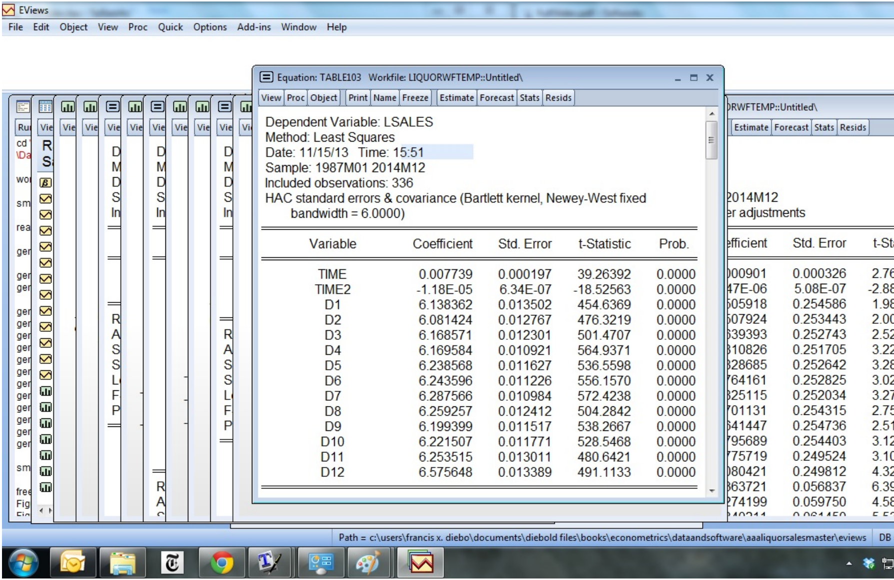
$$AR(1) = -3.797$$
$$AR(2) = -3.941$$
$$AR(3) = -4.080$$
$$AR(4) = -4.086$$
$$AR(5) = -4.071$$
$$AR(6) = -4.058$$
$$AR(7) = -4.057$$
$$AR(8) = -4.040$$

# Heteroskedasticity-and-Autocorrelation Consistent (HAC) Standard Errors

Using advanced methods, one can obtain consistent standard errors (if not an efficient $\hat{\beta}$), under minimal assumptions

- "HAC standard errors"
- "Robust standard errors"
- "Newey-West standard errors"
- $\hat{\beta}$ remains unchanged at its OLS value. Is that a problem?

# Trend + Seasonal Model with HAC Standard Errors



Equation: TABLE103 Workfile: LIQUORWFTEMP::Untitled\

Dependent Variable: LSALES
Method: Least Squares
Date: 11/15/13 Time: 15:51
Sample: 1987M01 2014M12
Included observations: 336
HAC standard errors & covariance (Bartlett kernel, Newey-West fixed bandwidth = 6.0000)

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| TIME | 0.007739 | 0.000197 | 39.26392 | 0.0000 |
| TIME2 | -1.18E-05 | 6.34E-07 | -18.52563 | 0.0000 |
| D1 | 6.138362 | 0.013502 | 454.6369 | 0.0000 |
| D2 | 6.081424 | 0.012767 | 476.3219 | 0.0000 |
| D3 | 6.168571 | 0.012301 | 501.4707 | 0.0000 |
| D4 | 6.169584 | 0.010921 | 564.9371 | 0.0000 |
| D5 | 6.238568 | 0.011627 | 536.5598 | 0.0000 |
| D6 | 6.243596 | 0.011226 | 556.1570 | 0.0000 |
| D7 | 6.287566 | 0.010984 | 572.4238 | 0.0000 |
| D8 | 6.259257 | 0.012412 | 504.2842 | 0.0000 |
| D9 | 6.199399 | 0.011517 | 538.2667 | 0.0000 |
| D10 | 6.221507 | 0.011771 | 528.5468 | 0.0000 |
| D11 | 6.253515 | 0.013011 | 480.6421 | 0.0000 |
| D12 | 6.575648 | 0.013389 | 491.1133 | 0.0000 |

# Structural Change

# Structural Change: Gradual

$$y_t = \beta_1 + \beta_{2t} x_t + \varepsilon_t$$

where

$$\beta_{1t} = \gamma_1 + \gamma_2 TIME_t$$

$$\beta_{2t} = \delta_1 + \delta_2 TIME_t$$

Then we have:

$$y_t = (\gamma_1 + \gamma_2 TIME_t) + (\delta_1 + \delta_2 TIME_t) x_t + \varepsilon_t$$

We simply run:

$$y_t \rightarrow c, \ , Time_t, \ x_t, \ TIME_t * x_t$$

This is yet another important use of dummies. The regression can be used both to test for structural change ($F$ test of $\gamma_2 = \delta_2 = 0$), and to accommodate it if present.

# Structural Change: Sharp Exogenous

$$y_t = \begin{cases} \beta_1^1 + \beta_2^1 x_t + \varepsilon_t, \ t = 1, ..., T^* \\ \beta_1^2 + \beta_2^2 x_t + \varepsilon_t, \ t = T^* + 1, ..., T \end{cases}$$

Let

$$D_t = \begin{cases} 0, \ t = 1, ..., T^* \\ D_t = 1, \ t = T^* + 1, ... T \end{cases}$$

Then we can write the model as:

$$y_t = (\beta_1^1 + (\beta_1^2 - \beta_1^1) D_t) + (\beta_2^1 + (\beta_2^2 - \beta_2^1) D_t) x_t + \varepsilon_t$$

We simply run:

$$y_t \to c, \ D_t, \ x_t, \ D_t \times x_t$$

# Structural Change: Sharp Exogenous, Continued

The regression can be used both to test for structural change, and to accommodate it if present. It represents yet another use of dummies. The no-break null corresponds to the joint hypothesis of zero coefficients on $D_t$ and $D_t \times x_t$, for which the "$F$" statistic is distributed $\chi^2$ asymptotically (and $F$ in finite samples under normality).

In the general case, under the no-break null the so-called Chow breakpoint test statistic,

$$Chow = \frac{(e'e - (e_1'e_1 + e_2'e_2))/K}{(e_1'e_1 + e_2'e_2)/(T-2K)},$$

is distributed $F$ in finite samples (under normality) and $\chi^2$ asymptotically.

# Structural Change: Sharp Endogenous

$$MaxChow = \max_{\tau_1 \leq \tau \leq \tau_2} Chow(\tau),$$

where $\tau$ denotes sample fraction
(typically we take $\tau_1 = .15$ and $\tau_2 = .85$).

The distribution of *MaxChow* has been tabulated.

# Recursive Estimation

$$y_t = \sum_{k=1}^{K} \beta_k x_{kt} + \varepsilon_t$$

$$\varepsilon_t \sim iidN(0, \sigma^2),$$

$$t = 1, ..., T.$$

OLS estimation uses the full sample, $t = 1, ..., T$.

Recursive least squares uses an expanding sample.
Begin with the first $K$ observations and estimate the model.
Then estimate using the first $K + 1$ observations, and so on.
At the end we have a set of recursive parameter estimates:
$\hat{\beta}_{k,t}$, for $k = 1, ..., K$ and $t = K, ..., T$.

# Recursive Residuals

At each $t$, $t = K, ..., T-1$, compute a 1-step forecast,

$$\hat{y}_{t+1,t} = \sum_{k=1}^{K} \hat{\beta}_{kt} x_{k,t+1}.$$

The corresponding forecast errors, or recursive residuals, are

$$\hat{e}_{t+1,t} = y_{t+1} - \hat{y}_{t+1,t}.$$

$$\hat{e}_{t+1,t} \sim N(0, \sigma^2 r_t)$$

where $r_t > 1$ for all $t$

# Standardized Recursive Residuals and CUSUM

$$w_{t+1,t} \equiv \frac{\hat{e}_{t+1,t}}{\sigma\sqrt{r}_t},$$

$$t = K, ..., T-1.$$

Under the maintained assumptions,

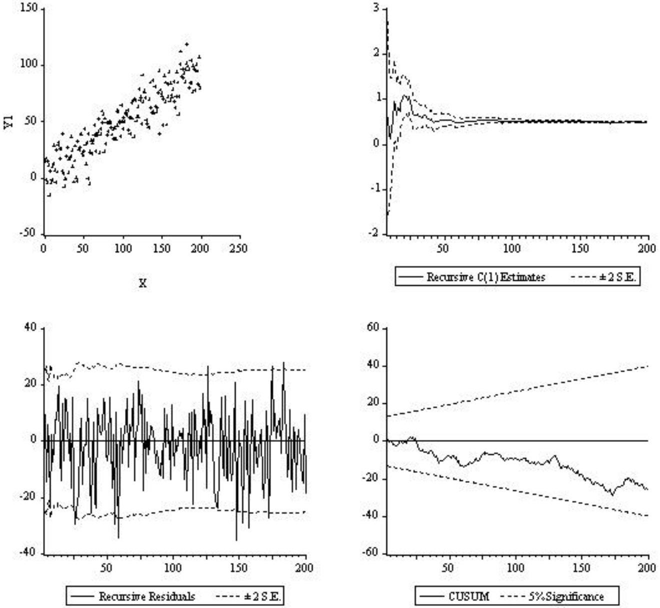
$$w_{t+1,t} \sim iidN(0,1).$$

Then

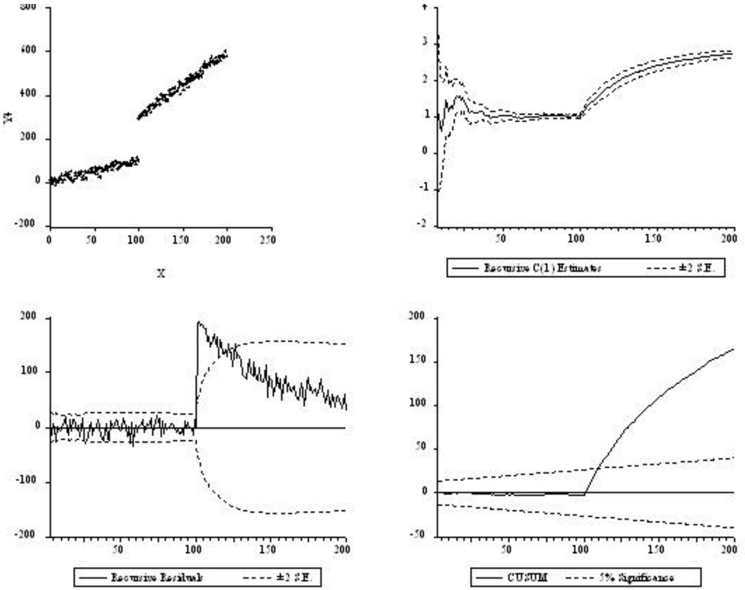
$$CUSUM_{t*} \equiv \sum_{t=K}^{t^*} w_{t+1,t}, \;\; t^* = K, ..., T-1$$

is just a sum of *iid* $N(0,1)$'s (i.e. a Gaussian random walk).

# Recursive Analysis, Constant Parameter



# Recursive Analysis, Breaking Parameter



# Regime Switching I: Observed-Regime Threshold Model

$$
y_t = \begin{cases} c^{(u)} + \phi^{(u)} y_{t-1} + \varepsilon_t^{(u)}, & \theta^{(u)} < y_{t-d} \\ c^{(m)} + \phi^{(m)} y_{t-1} + \varepsilon_t^{(m)}, & \theta^{(l)} < y_{t-d} < \theta^{(u)} \\ c^{(l)} + \phi^{(l)} y_{t-1} + \varepsilon_t^{(l)}, & \theta^{(l)} > y_{t-d} \end{cases}
$$

# Regime Switching II: Markov-Switching Model

Regime governed by latent 2-state Markov process:

$$M = \begin{pmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{pmatrix}$$

Switching mean:

$$f(y_t | s_t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_t - \mu_{s_t})^2}{2\sigma^2}\right).$$

Switching regression:

$$f(y_t | s_t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_t - x_t'\beta_{s_t})^2}{2\sigma^2}\right).$$

# Rolling Regression for Generic Structural Change

# Heteroskedasticity in Time Series

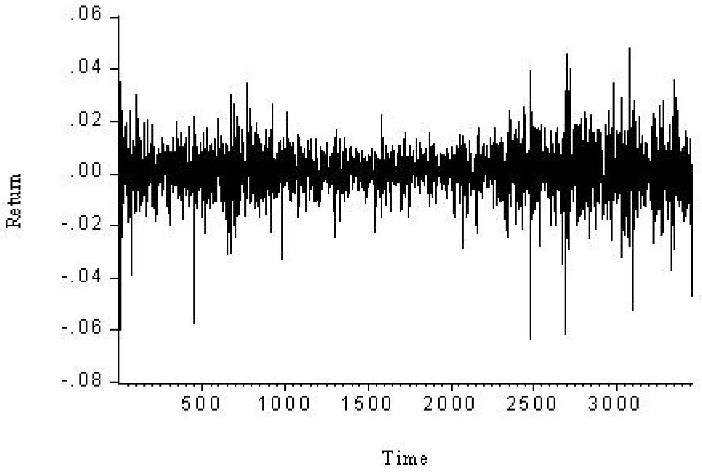



Figure: Time Series of Daily NYSE Returns.

# Key Fact 1: Stock Returns are Approximately Serially Uncorrelated

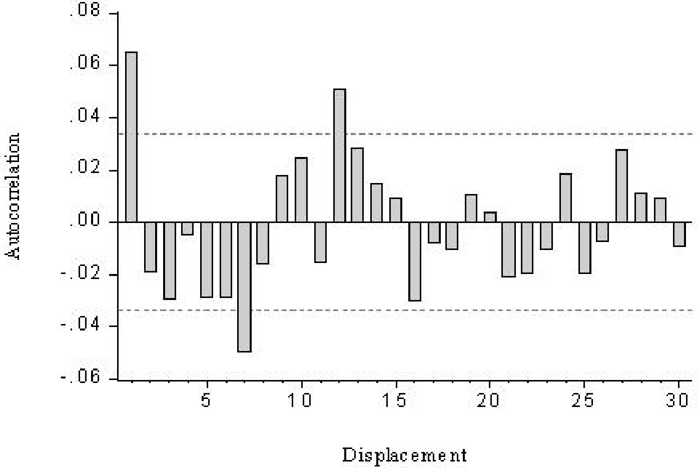



Figure: Correlogram of Daily Stock Market Returns.

# Key Fact 2: Returns are Unconditionally Non-Gaussian

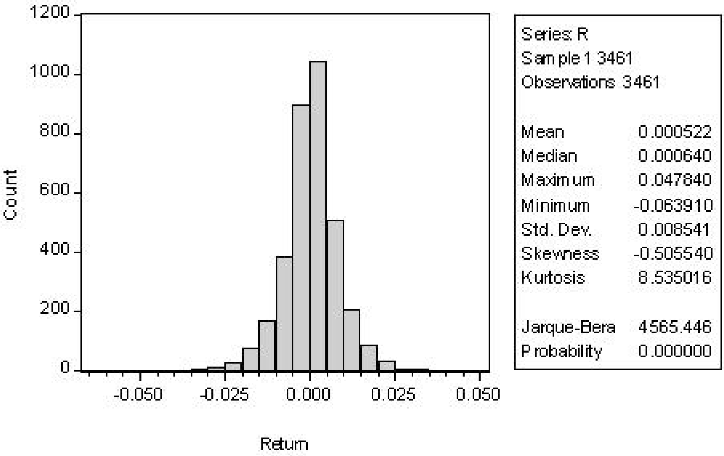



Figure: Histogram and Statistics for Daily NYSE Returns.

# Unconditional Volatility Measures

Variance: $\sigma^2 \ = \ E(r_t - \mu)^2$ (or standard deviation: $\sigma$)

Mean Absolute Deviation: $MAD \ = \ E|r_t - \mu|$

Interquartile Range: $IQR \ = \ 75\% \ - \ 25\%$

Outlier probability: $P|r_t - \mu| \ > \ 5\sigma$ (for example)

Tail index: $\gamma$ s.t. $P(r_t > r) \ = \ k \ r^{-\gamma}$

Kurtosis: $K \ = \ E(r - \mu)^4/\sigma^4$

$p$% Value at Risk ($VaR^p$)): $x$ s.t. $P(r_t < x) = p$

# Key Fact 3: Returns are Conditionally Heteroskedastic I

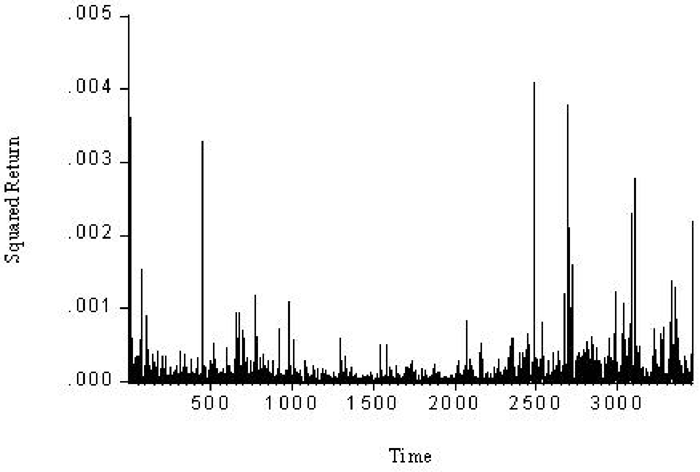



Figure: Time Series of Daily Squared NYSE Returns

# Key Fact 3: Returns are Conditionally Heteroskedastic II

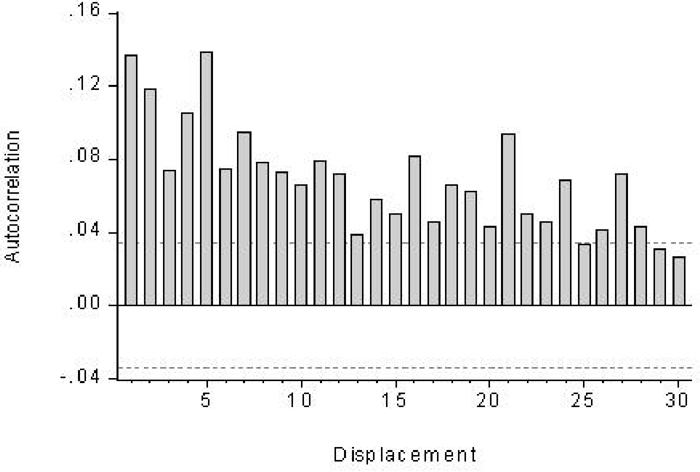



Figure: Correlogram of Daily Squared NYSE Returns.

# Background: Financial Economics Changes Fundmentally When Volatility is Dynamic

- Risk management
- Portfolio allocation
- Asset pricing
- Hedging
- Trading

# Asset Pricing I: Sharpe Ratios

Standard Sharpe:

$$\frac{E(r_{it} - r_{ft})}{\sigma}$$

Conditional Sharpe:

$$\frac{E(r_{it} - r_{ft})}{\sigma_t}$$

# Asset Pricing II: CAPM

Standard CAPM:

$$(r_{it} - r_{ft}) = \alpha + \beta(r_{mt} - r_{ft})$$

$$\beta \;=\; \frac{cov((r_{it} - r_{ft}),\ (r_{mt} - r_{ft}))}{var(r_{mt} - r_{ft})}$$

Conditional CAPM:

$$\beta_t \;=\; \frac{cov_t((r_{it} - r_{ft}),\ (r_{mt} - r_{ft}))}{var_t(r_{mt} - r_{ft})}$$

# Asset Pricing III: Derivatives

Black-Scholes:

$$C = N(d_1)S - N(d_2)Ke^{-r\tau}$$

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

$$P_C \; = \; BS(\sigma, \; ...)$$

(Standard Black-Scholes options pricing)

Completely different when $\sigma$ varies!

# Conditional Return Distributions

$$f(r_t) \text{ } vs. \text{ } f(r_t|\Omega_{t-1})$$

Key 1: $E(r_t|\Omega_{t-1})$

Are returns conditional mean independent? Arguably yes.

Returns are (arguably) approximately serially uncorrelated, and (arguably) approximately free of additional non-linear conditional mean dependence.

# Conditional Return Distributions, Continued

Key 2: $var(r_t|\Omega_{t-1}) \;=\; E((r_t-\mu)^2|\Omega_{t-1})$

Are returns conditional variance independent? No way!

Squared returns serially correlated, often with very slow decay.

# Linear Models (e.g., $AR(1)$)

$$r_t = \phi r_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim iid(0,\ \sigma^2), \quad |\phi| < 1$$

Uncond. mean: $E(r_t) = 0$ (constant)
Uncond. variance: $E(r_t^2) = \sigma^2/(1-\phi^2)$ (constant)
Cond. mean: $E(r_t \mid \Omega_{t-1}) = \phi r_{t-1}$ (varies)
Cond. variance: $E([r_t - E(r_t \mid \Omega_{t-1})]^2 \mid \Omega_{t-1}) = \sigma^2$ (constant)

– Conditional mean adapts, but conditional variance does not

# ARCH(1) Process

$$r_t|\Omega_{t-1} \sim N(0, h_t)$$

$$h_t = \omega + \alpha r_{t-1}^2$$

$$E(r_t) = 0$$

$$E({r_t}^2) = \frac{\omega}{(1-\alpha)}$$

$$E(r_t|\Omega_{t-1}) = 0$$

$$E([r_t - E(r_t|\Omega_{t-1})]^2|\Omega_{t-1}) = \omega + \alpha r_{t-1}^2$$

# GARCH(1,1) Process ("Generalized ARCH")

$$r_t \mid \Omega_{t-1} \sim N(0, h_t)$$

$$h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}$$

$$E(r_t) = 0$$

$$E({r_t}^2) = \frac{\omega}{(1 - \alpha - \beta)}$$

$$E(r_t|\Omega_{t-1}) = 0$$

$$E([r_t - E(r_t \mid \Omega_{t-1})]^2 \mid \Omega_{t-1}) = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}$$

Well-defined and covariance stationary if
$0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta < 1$

# GARCH(1,1) and Exponential Smoothing

Exponential smoothing recursion:

$$\hat{\sigma}_t^2 = \lambda \hat{\sigma}_{t-1}^2 + (1-\lambda) r_t^2$$

$$\implies \hat{\sigma}_t^2 = (1-\lambda) \sum_j \lambda^j r_{t-j}^2$$

But in GARCH(1,1) we have:

$$h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}$$

$$h_t = \frac{\omega}{1-\beta} + \alpha \sum \beta^{j-1} r_{t-j}^2$$

# Unified Theoretical Framework

- Volatility dynamics (of course, by construction)
- Volatility clustering produces unconditional leptokurtosis
- Temporal aggregation reduces the leptokurtosis

# Tractable Empirical Framework

$$L(\theta; r_1, \ldots, r_T) = f(r_T|\Omega_{T-1}; \theta) f((r_{T-1}|\Omega_{T-2}; \theta) \ldots,$$

where $\theta = (\omega, \alpha, \beta)'$

If the conditional densities are Gaussian,

$$f(r_t|\Omega_{t-1}; \theta) = \frac{1}{\sqrt{2\pi}} h_t(\theta)^{-1/2} \exp\left(-\frac{1}{2}\frac{r_t^2}{h_t(\theta)}\right),$$

so

$$\ln L = const - \frac{1}{2}\sum_t \ln h_t(\theta) - \frac{1}{2}\sum_t \frac{r_t^2}{h_t(\theta)}$$

# Variations on the GARCH Theme

- Explanatory variables in the variance equation: GARCH-X
- Fat-tailed conditional densities: t-GARCH
- Asymmetric response and the leverage effect: T-GARCH
- Regression with GARCH disturbances
- Time-varying risk premia: GARCH-M

# Explanatory variables in the Variance Equation: GARCH-X

$$h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} + \gamma z_t$$

where $z$ is a positive explanatory variable

# Fat-Tailed Conditional Densities: t-GARCH

If $r$ is conditionally Gaussian, then

$$r_t = \sqrt{h_t}\; N(0,1)$$

But often with high-frequency data,

$$\frac{r_t}{\sqrt{h_t}} \sim leptokurtic$$

So take:

$$r_t = \sqrt{h_t}\; \frac{t_d}{std(t_d)}$$

and treat $d$ as another parameter to be estimated

# Asymmetric Response and the Leverage Effect: T-GARCH

Standard GARCH: $h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}$

T-GARCH: $h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 D_{t-1} + \beta h_{t-1}$

$$D_t = \begin{cases} 1 \; if \;\; r_t < 0 \\ 0 \; otherwise \end{cases}$$

positive return (good news): $\alpha$ effect on volatility

negative return (bad news): $\alpha + \gamma$ effect on volatility

$\gamma \neq 0$: Asymetric news response
$\gamma > 0$: "Leverage effect"

# Regression with GARCH Disturbances

$$y_t = x_t'\beta + \varepsilon_t$$

$$\varepsilon_t | \Omega_{t-1} \sim N(0, h_t)$$

# Time-Varying Risk Premia: GARCH-M

Standard GARCH regression model:

$$y_t = x_t'\beta + \varepsilon_t$$

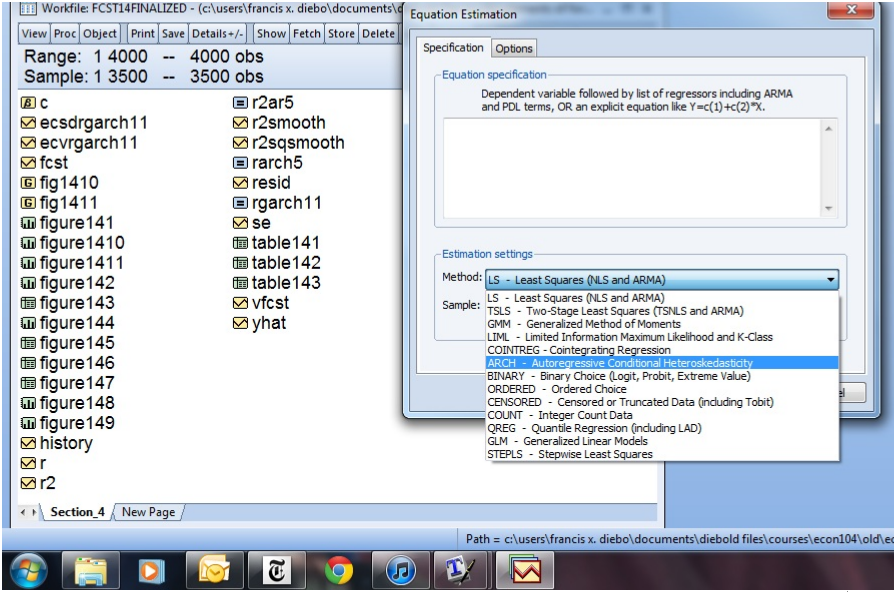
$$\varepsilon_t | \Omega_{t-1} \sim N(0, h_t)$$

GARCH-M model is a special case:

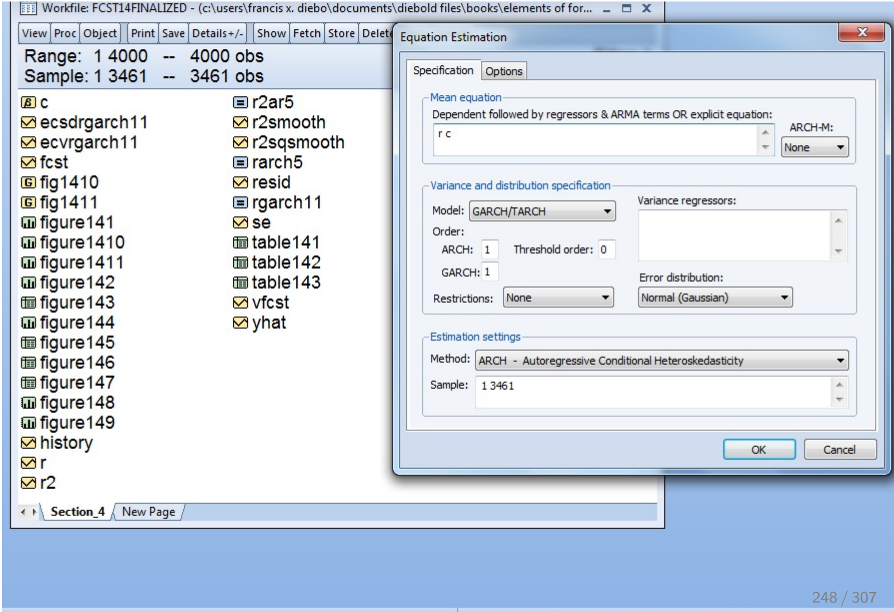
$$y_t = x_t'\beta + \gamma h_t + \varepsilon_t$$

$$\varepsilon_t | \Omega_{t-1} \sim N(0, h_t)$$

# Back to Empirical Work – “Standard” GARCH(1,1)



# GARCH(1,1)



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Range: 1 4000 -- 4000 obs
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figure149
history
r
r2
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r2smooth
r2sqsmooth
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resid
rgarch11
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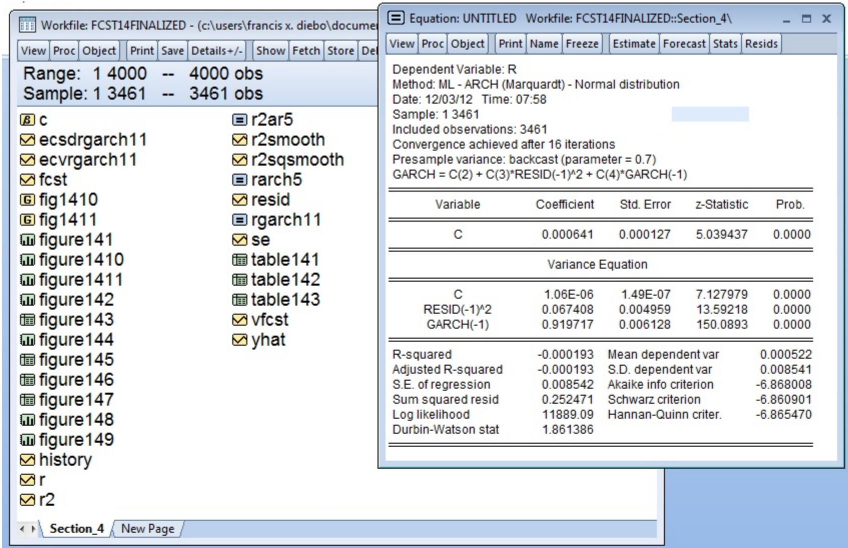
table141
table142
table143
vfcst
yhat
Section_4
New Page
Equation Estimation
Specification
Options
Mean equation
Dependent followed by regressors & ARMA terms OR explicit equation:
r c
ARCH-M:
None
Variance and distribution specification
Model:
GARCH/TARCH
Order:
ARCH: 1
Threshold order: 0
GARCH: 1
Restrictions:
None
Variance regressors:
Error distribution:
Normal (Gaussian)
Estimation settings
Method:
ARCH - Autoregressive Conditional Heteroskedasticity
Sample:
1 3461
OK
Cancel

# GARCH(1,1)



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New Page
Equation: UNTITLED Workfile: FCST14FINALIZED::Section_4\
View Proc Object Print Name Freeze Estimate Forecast Stats Resids
Dependent Variable: R
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 12/03/12 Time: 07:58
Sample: 1 3461
Included observations: 3461
Convergence achieved after 16 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)
Variable Coefficient Std. Error z-Statistic Prob.
C 0.000641 0.000127 5.039437 0.0000
Variance Equation
C 1.06E-06 1.49E-07 7.127979 0.0000
RESID(-1)^2 0.067408 0.004959 13.59218 0.0000
GARCH(-1) 0.919717 0.006128 150.0893 0.0000
R-squared -0.000193 Mean dependent var 0.000522
Adjusted R-squared -0.000193 S.D. dependent var 0.008541
S.E. of regression 0.008542 Akaike info criterion -6.868008
Sum squared resid 0.252471 Schwarz criterion -6.860901
Log likelihood 11889.09 Hannan-Quinn criter. -6.865470
Durbin-Watson stat 1.861386

# GARCH(1,1)

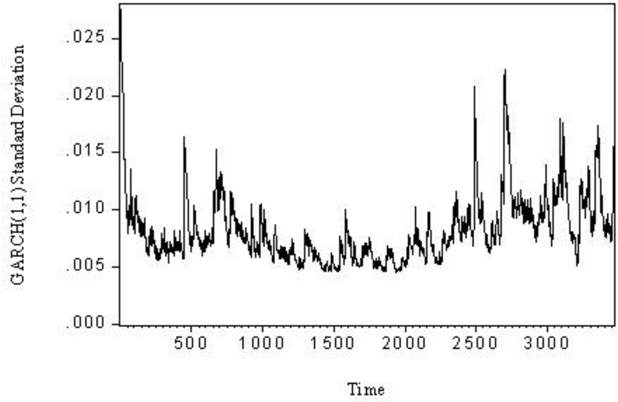



Figure: Estimated Conditional Standard Deviation, Daily NYSE Returns.

# GARCH(1,1)

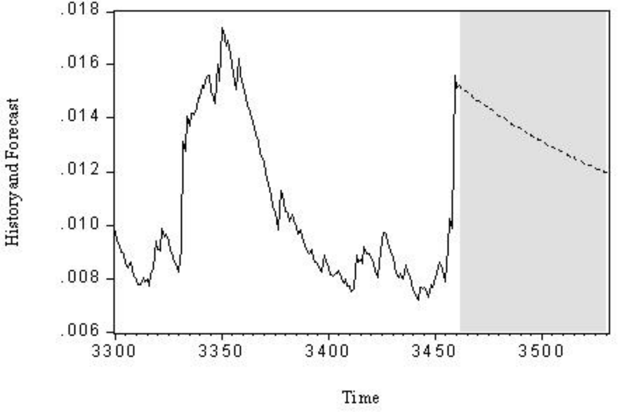



Figure: Conditional Standard Deviation, History and Forecast, Daily NYSE Returns.

# A Useful Specification Diagnostic

$$r_t | \Omega_{t-1} \sim N(0, h_t)$$

$$r_t = \sqrt{h_t}\varepsilon_t, \quad \varepsilon_t \sim iidN(0,1)$$

$$\frac{r_t}{\sqrt{h_t}} = \varepsilon_t, \quad \varepsilon_t \sim iidN(0,1)$$

Infeasible: examine $\varepsilon_t$. iid? Gaussian?

Feasible: examine $\hat{\varepsilon}_t = r_t / \sqrt{\hat{h}_t}$. iid? Gaussian?

Key deviation from iid is volatility dynamics. So examine correlogram of squared standardized returns, $\hat{\varepsilon}_t^2$

# GARCH(1,1)

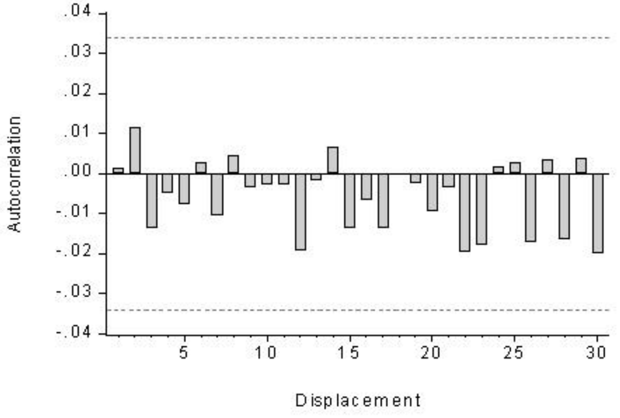
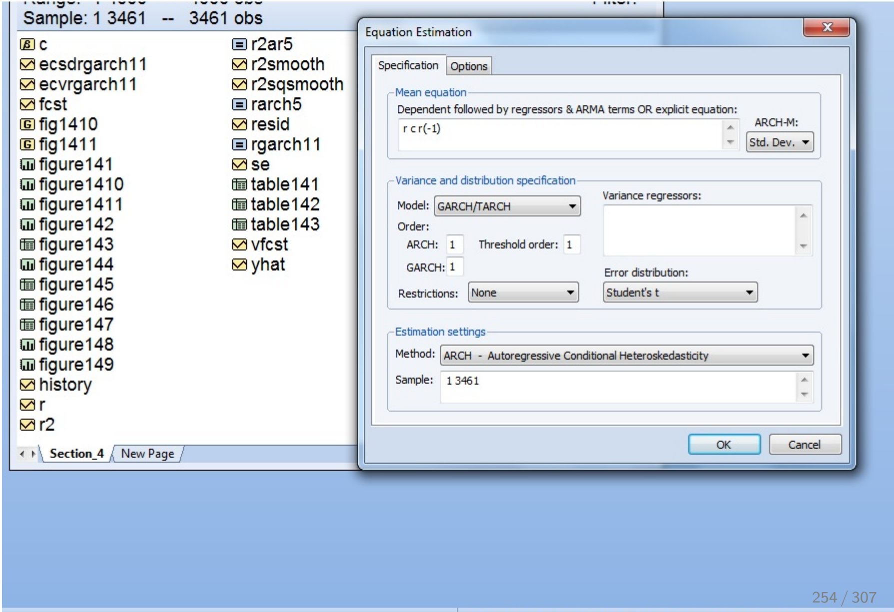



Figure: Correlogram of Squared Standardized GARCH(1,1) Residuals, Daily NYSE Returns.

# "Fancy" GARCH(1,1)



# "Fancy" GARCH(1,1)

Dependent Variable: R
Method: ML - ARCH (Marquardt) - Student's t distribution
Date: 04/10/12 Time: 13:48
Sample (adjusted): 2 3461
Included observations: 3460 after adjustments
Convergence achieved after 19 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(4) + C(5)*RESID(-1)^2 + C(6)*RESID(-1)^2*(RESID(-1)<0)
+ C(7)*GARCH(-1)

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| @SQRT(GARCH) | 0.083360 | 0.053138 | 1.568753 | 0.1167 |
| C | 1.28E-05 | 0.000372 | 0.034443 | 0.9725 |
| R(-1) | 0.073763 | 0.017611 | 4.188535 | 0.0000 |
| Variance Equation | | | | |
| C | 1.03E-06 | 2.23E-07 | 4.628790 | 0.0000 |
| RESID(-1)^2 | 0.014945 | 0.009765 | 1.530473 | 0.1259 |
| RESID(-1)^2*(RESID(-1)<0) | 0.094014 | 0.014945 | 6.290700 | 0.0000 |
| GARCH(-1) | 0.922745 | 0.009129 | 101.0741 | 0.0000 |
| T-DIST. DOF | 5.531579 | 0.478432 | 11.56188 | 0.0000 |

# Causal Predictive Modeling

Consider a standard linear regression setting with $K$ regressors and sample size $N$.

# T-Consistency

We will say that an estimator $\hat{\beta}$ is *consistent for a treatment effect* ("T-consistent") if $plim \hat{\beta}_k = \partial E(y|x)/\partial x_k$, $\forall k = 1, ..., K$; that is, if

$$\left( \hat{\beta}_k - \frac{\partial E(y|x)}{\partial x_k} \right) \rightarrow_p 0, \ \forall k = 1, ..., K.$$

Hence in large samples $\hat{\beta}_k$ provides a good estimate of the effect on $y$ of a one-unit "treatment" or "intervention" performed on $x_k$.
T-consistency is the standard econometric notion of consistency.
OLS is T-consistent under the FIC.
*OLS is generally* **not** *T-consistent without the FIC.*

# And Remember How Stringent the FIC *Are*!

1. The fitted model is:

$$y = X\beta + \varepsilon$$

$$\varepsilon \sim N(\underline{0}, \sigma^2 I),$$

and it matches the true data-generating process.

   1.1 The relationship, if any, is truly linear, with no omitted variables, no measurement error, etc.
   1.2 The coefficients, $\beta$, are fixed.
   1.3 $\varepsilon \sim N$.
   1.4 The $\varepsilon$'s have constant variance $\sigma^2$.
   1.5 The $\varepsilon$'s are uncorrelated.

2. There is no redundancy among the variables contained in $X$, so that $X'X$ is non-singular.
3. $X$ is a non-stochastic matrix, fixed in repeated samples (old style), or $X$ is a stochastic matrix such that $E(\varepsilon|X) = 0$ (new style).

# Non-Causal Predictive Modeling

Again consider a standard linear regression setting with $K$ regressors and sample size $N$.

# P-Consistency

Assuming quadratic loss,
the predictive risk of a parameter configuration $\beta$ is

$$R(\beta) = E(y - x'\beta)^2.$$

Let $B$ be a set of $\beta$'s and let $\beta^* \in B$ minimize $R(\beta)$.

We will say that $\hat{\beta}$ is *consistent for a predictive effect* ("P-consistent") if $plim R(\hat{\beta}) = R(\beta^*)$; that is, if

$$\left(R(\hat{\beta}) - R(\beta^*)\right) \to_p 0.$$

Hence in large samples $\hat{\beta}$ provides a good way to predict $y$ for any hypothetical $x$: simply use $x'\hat{\beta}$.
*OLS is effectively* **always** *P-consistent;*
*we require almost no conditions of any kind!*

# Correlation vs. Causality, and P-Consistency vs. T-consistency

The distinction between P-consistency and T-consistency is related to the distinction between correlation and causality. As is well known, correlation does not imply causality! As long as $x$ and $y$ are correlated, we can exploit the correlation (as captured in $\hat{\beta}$) very generally to predict $y$ given knowledge of $x$. That is, there will be a nonzero "predictive effect" of $x$ knowledge on $y$. But nonzero correlation doesn't necessarily tell us anything about the causal "treatment effect" of $x$ *treatments* on $y$. That requires the full ideal conditions. Even if there is a non-zero predictive effect of $x$ on $y$ (as captured by $\hat{\beta}_{LS}$), there may or may not be a nonzero treatment effect of $x$ on $y$, and even if nonzero it will generally not equal the predictive effect.

# Correlation vs. Causality, and P-Consistency vs. T-consistency, Continued

So, assembling things:

P-consistency is consistency for a non-causal predictive effect.
(Almost trivially easy to obtain.)

T-consistency is consistency for a causal predictive effect.
(Notoriously difficult to obtain.)

# An Example of Correlation Without Causality

To take a simple example, suppose that $y$ and $x$ are in fact causally *un*related, so that the true treatment effect of $x$ on $y$ is 0 by construction. But suppose that $x$ is also highly correlated with an unobserved variable $z$ that *does* cause $y$. Then $y$ and $x$ will be correlated due to their joint dependence on $z$, and that correlation can be used to predict $y$ given $x$, despite the fact that, by construction, $x$ treatments (interventions) will have no effect on $y$.

# A Thought Experiment

True DGP:

$$y_i = z_i + \varepsilon_i$$

Suppose also that there exists a variable $x$ such that $corr(x, z) > 0$.

Fitted OLS Regression Model:

$$y \rightarrow x$$

Is $\hat{\beta}_{OLS}$ P-consistent?

Is $\hat{\beta}_{OLS}$ T-consistent?

# Nonstationarity

# Nonstationarity and Random Walks

Random walk:

$$y_t = y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim iid(0, \sigma^2)$$

Just a simple special case of $AR(1)$ $\phi = 1$

# Random Walk with Drift

$$y_t = \delta + y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim iid(0, \sigma^2)$$

$$y_t = t\delta + y_0 + \sum_{i=1}^{t} \varepsilon_i$$

$$E(y_t) = y_0 + t\delta$$

$$var(y_t) = t\sigma^2$$

$$\lim_{t \to \infty} var(y_t) = \infty$$

# Recall Properties of $AR(1)$ with $|\phi| < 1$

– Shocks $\varepsilon_t$ have persistent but not permanent effects

$$y_t = \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-j} \quad (\text{note } \phi_j \to 0)$$

– Series $y_t$ varies but not too extremely

$$var(y_t) = \frac{\sigma^2}{1-\phi^2} \quad (\text{note } var(y_t) < \infty)$$

– Autocorrelations $\rho(\tau)$ nonzero but decay to zero

$$\rho(\tau) = \phi^\tau \quad (\text{note } \phi^\tau \to 0)$$

# Properties of the Random Walk ($AR(1)$ With $|\phi| = 1$)

– Shocks have permanent effects

$$y_t = y_0 + \sum_{j=0}^{t-1} \varepsilon_{t-j}$$

– Series is infinitely variable

$$E(y_t) = y_0$$

$$var(y_t) = t\sigma^2$$

$$\lim_{t\to\infty} var(y_t) = \infty$$

– Autocorrelations $\rho(\tau)$ do not decay

$$\rho(\tau) \approx 1 \;\; \text{(formally not defined)}$$

# A Key Insight Regarding the Random Walk

– Level series $y_t$ is non-stationary (of course)

– Differenced series $y_t$ *is* stationary (indeed white noise)!

$$\Delta y_t = \varepsilon_t$$

A series is called $I(d)$ if it is non-stationary in levels but is appropriately made stationary by differencing $d$ times.

Random walk is the key $I(1)$ process.
Other $I(1)$ processes are similar. Why?

# The Beveridge-Nelson Decomposition

$$y_t \sim I(1) \implies y_t = x_t + z_t$$
$$x_t = \text{random walk}$$
$$z_t = \text{covariance stationary}$$

Hence the random walk is the key ingredient for all $I(1)$ processes.

The Beveridge-Nelson decomposition implies that shocks to any $I(1)$ process have some permanent effect, as with a random walk. But the effects are not *completely* permanent, unless the process is a pure random walk.

# $I(1)$ Processes and "Unit Roots"

Random walk is an $I(1)$ $AR(1)$ process:

$$y_t = y_{t-1} + \varepsilon_t$$

$$\underbrace{(1-L)}_{\text{deg } 1} y_t = \varepsilon_t$$

One (unit) root, $L = 1$

$\Delta y_t$ is standard covariance-stationary $WN$

More general $I(1)$ $AR(p)$ process:

$$\underbrace{\Phi(L)}_{\text{deg p}} y_t = \varepsilon_t$$

$$[\underbrace{\Phi'(L)}_{\text{(deg p-1)}} \underbrace{(1-L)}_{\text{(deg 1)}}] y_t = \varepsilon_t$$

$p - 1$ stationary roots, one unit root

$\Delta y_t$ is standard covariance stationary $AR(p-1)$

# Unit Root Distribution for the AR(1) Process

Key issue (hypothesis) in economics:

$I(1)$ vs. $I(0)$, unit root vs. stationary process

When $|\phi| < 1$,

$$\sqrt{T}(\hat{\phi}_{LS} - \phi) \overset{d}{\rightarrow} N$$

When $\phi = 1$,

$$T(\hat{\phi}_{LS} - 1) \overset{d}{\rightarrow} DF$$

Superconsistent

Nonstandard limiting distribution

Downward finite-sample bias ("Dickey-Fuller bias")

# Studentized Statistic

$$\hat{\tau} = \frac{\hat{\phi} - 1}{s\sqrt{\frac{1}{\sum y_{t-1}^2}}}$$

Not $t$ in finite samples
Not $N(0,1)$ asymptotically

Trick:
Don't run $y_t \to y_{t-1}$
Instead run $\Delta y_t \to y_{t-1}$

# $AR(1)$ With Nonzero Mean Under the Alternative

$$(y_t - \mu) = \phi(y_{t-1} - \mu) + \varepsilon_t$$

$$y_t = \alpha + \phi y_{t-1} + \varepsilon_t$$

where $\alpha = \mu(1 - \phi)$

Random walk null vs. mean-reverting alternative

Studentized statistic $\hat{\tau}_\mu$

# $AR(1)$ With Trend Under the Alternative

$$(y_t - a - bt) = \phi(y_{t-1} - a - b(t-1)) + \varepsilon_t$$

$$y_t = \alpha + \beta t + \phi y_{t-1} + \varepsilon_t$$

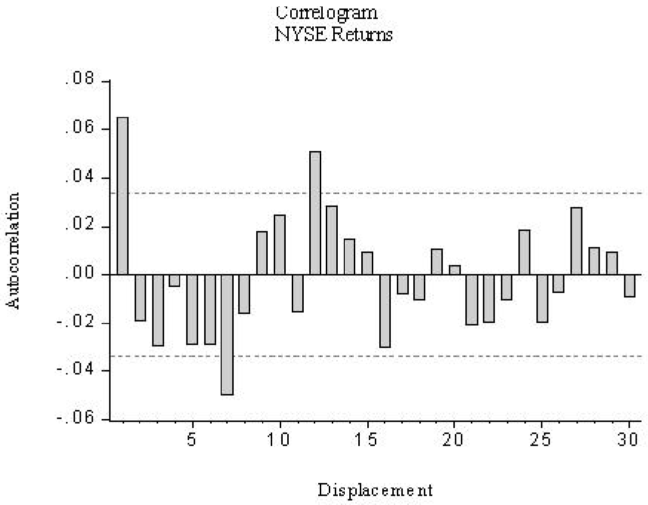
where $\alpha = a(1-\phi) + b\phi$ and $\beta = b(1-\phi)$

$H_0 : \phi = 1$ (unit root)
$H_1 : \phi < 1$ (stationary root)

Studentized statistic $\hat{\tau}_\tau$

"Random walk with drift" vs. "stat. $AR(1)$ around linear trend"
"Stochastic trend" vs. "deterministic trend"

# Stochastic Trend vs. Deterministic Trend



# $AR(p)$

$$y_t + \sum_{j=1}^{p} \phi_j y_{t-j} = \varepsilon_t$$

$$y_t = \rho_1 y_{t-1} + \sum_{j=2}^{p} \rho_j (y_{t-j+1} - y_{t-j}) + \varepsilon_t$$

where $p \geq 2$, $\rho_1 = -\sum_{j=1}^{p} \phi_j$, and $\rho_i = \sum_{j=i}^{p} \phi_j$, $i = 2, ..., p$

Studentized statistic $\hat{\tau}$ is still relevant

# $AR(p)$ With Nonzero Mean Under the Alternative

$$(y_t - \mu) + \sum_{j=1}^{p} \phi_j (y_{t-j} - \mu) = \varepsilon_t$$

$$y_t = \alpha + \rho_1 y_{t-1} + \sum_{j=2}^{p} \rho_j (y_{t-j+1} - y_{t-j}) + \varepsilon_t$$

where $\alpha = \mu(1 + \sum_{j=1}^{p} \phi_j)$

Studentized statistic $\hat{\tau}_\mu$ is still relevant

# $AR(p)$ With Trend Under the Alternative

$$(y_t - a - bt) + \sum_{j=1}^{p} \phi_j (y_{t-j} - a - b(t-j)) = \varepsilon_t$$

$$y_t = k_1 + k_2\ t + \rho_1 y_{t-1} + \sum_{j=2}^{p} \rho_j (y_{t-j+1} - y_{t-j}) + \varepsilon_t$$

$$k_1 = a\left(1 + \sum_{i=1}^{p} \phi_i\right) - b \sum_{i=1}^{p} i\phi_i$$

$$k_2 = b\left(1 + \sum_{i=1}^{p} \phi_i\right)$$

Under the null hypothesis, $k_1 = -b \sum_{i=1}^{p} i\phi_i$ and $k_2 = 0$

Studentized statistic $\hat{\tau}_\tau$ is still relevant

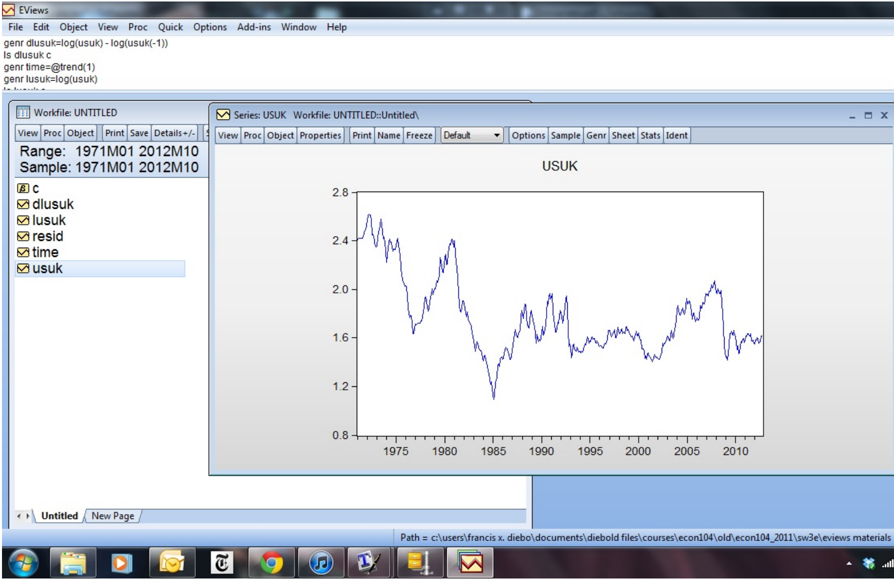
# "Trick Form" of ADF in the General $AR(p)$ Case

$$(y_t - y_{t-1}) = (\rho_1 - 1)y_{t-1} + \sum_{j=2}^{k-1} \rho_j (y_{t-j+1} - y_{t-j}) + \varepsilon_t$$

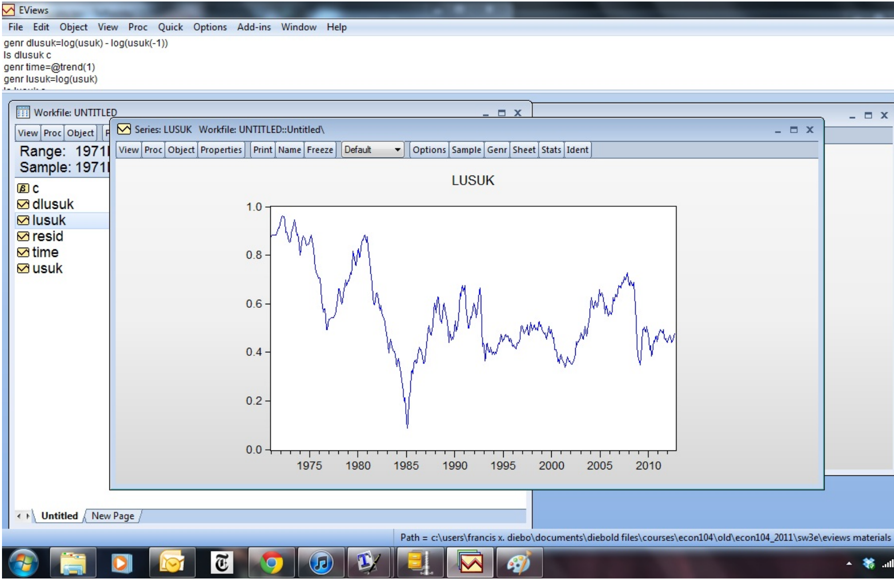
– Unit root corresponds to $(\rho_1 - 1) = 0$

– Use standard automatically-computed $t$-statistic (which of course does not have the $t$-distribution)

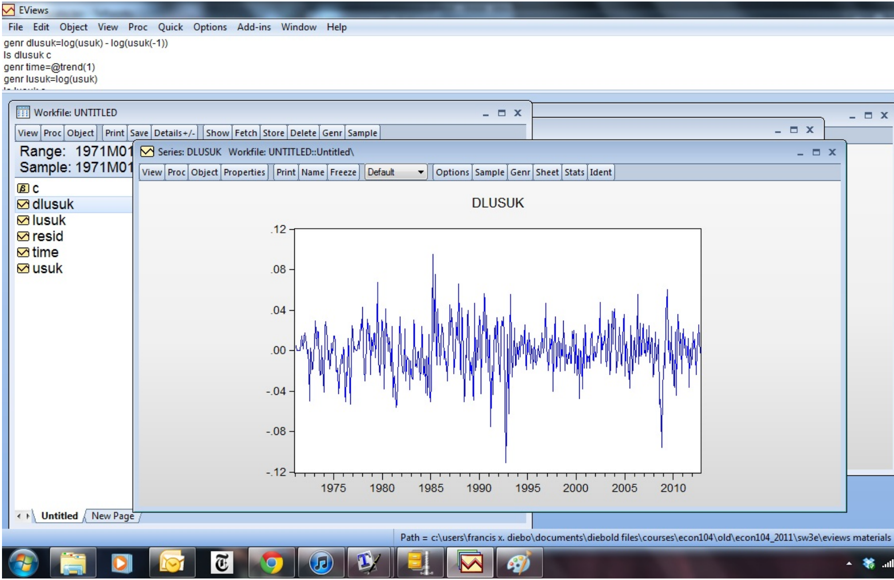
# USD/GBP Exchange Rate, 1971.01-2012.10



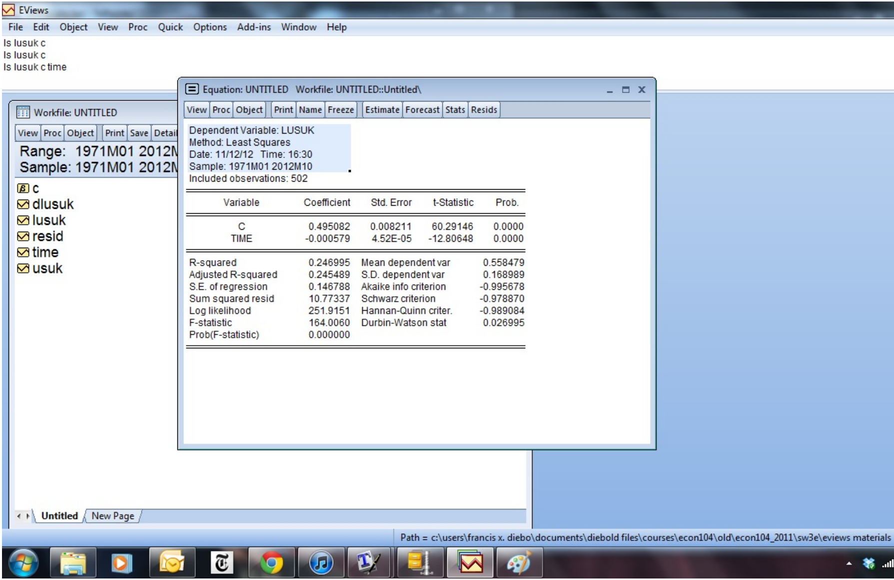
# Log USD/GBP Exchange Rate, 1971.01-2012.10



# Change in USD/GBP Exchange Rate, 1971.01-2012.10



# Trend-Stationary Model

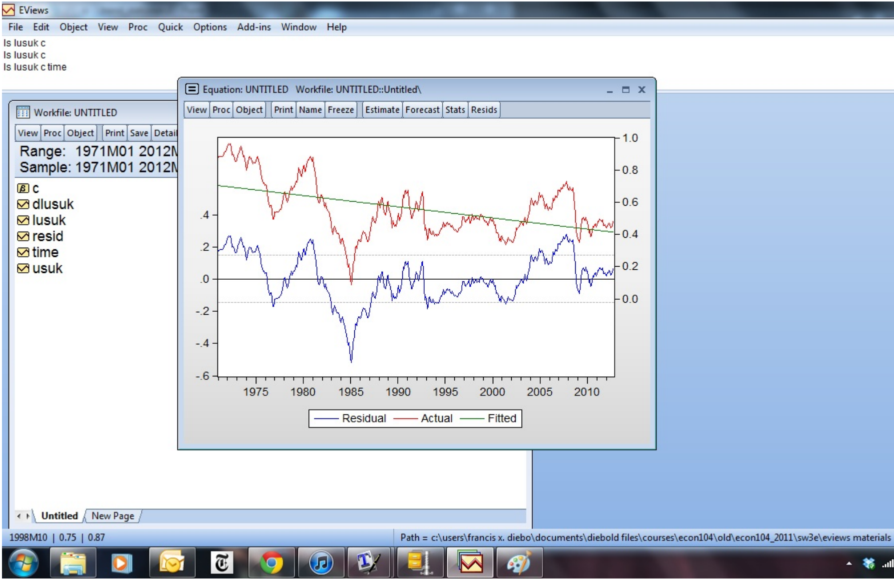



Dependent Variable: LUSUK
Method: Least Squares
Date: 11/12/12 Time: 16:30
Sample: 1971M01 2012M10
Included observations: 502

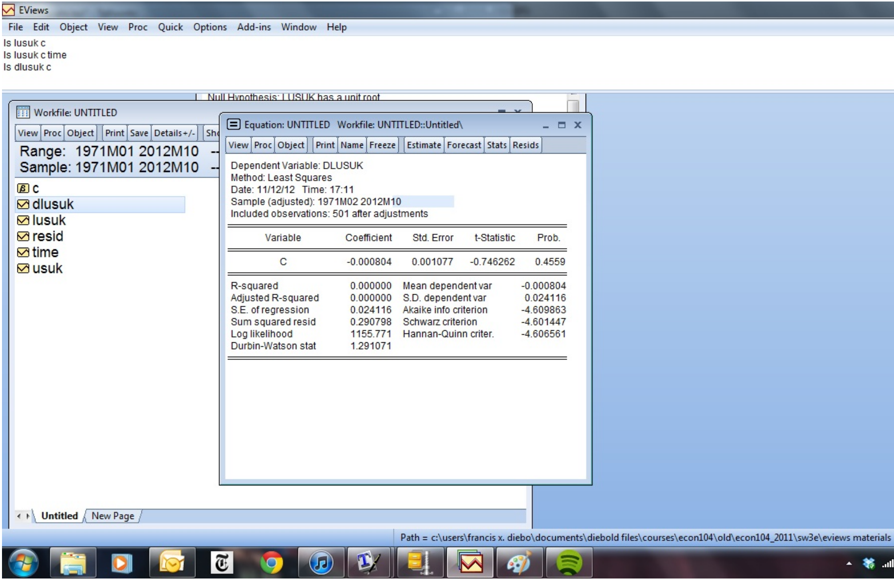
| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.495082 | 0.008211 | 60.29146 | 0.0000 |
| TIME | -0.000579 | 4.52E-05 | -12.80648 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.246995 | Mean dependent var | 0.558479 |
| Adjusted R-squared | 0.245489 | S.D. dependent var | 0.168989 |
| S.E. of regression | 0.146788 | Akaike info criterion | -0.995678 |
| Sum squared resid | 10.77337 | Schwarz criterion | -0.978870 |
| Log likelihood | 251.9151 | Hannan-Quinn criter. | -0.989084 |
| F-statistic | 164.0060 | Durbin-Watson stat | 0.026995 |
| Prob(F-statistic) | 0.000000 | | |

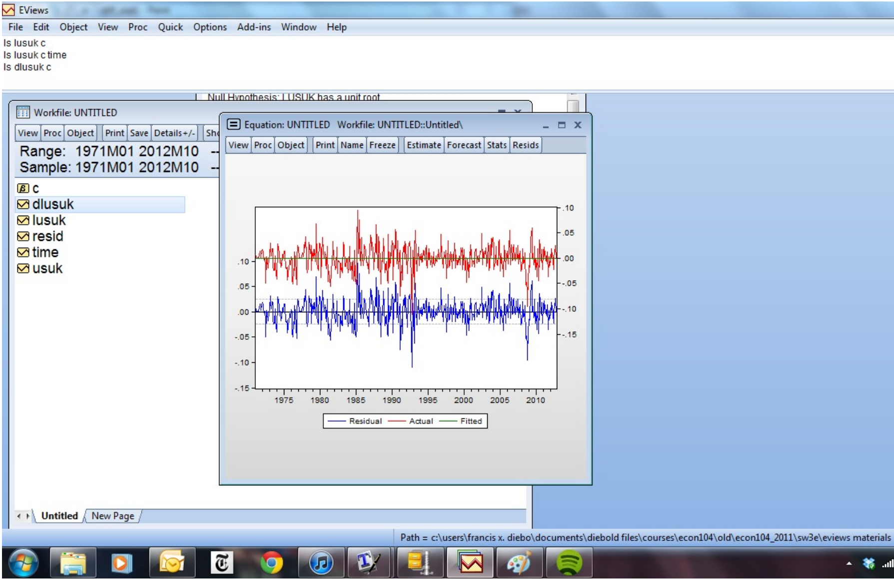
# Trend-Stationary Model



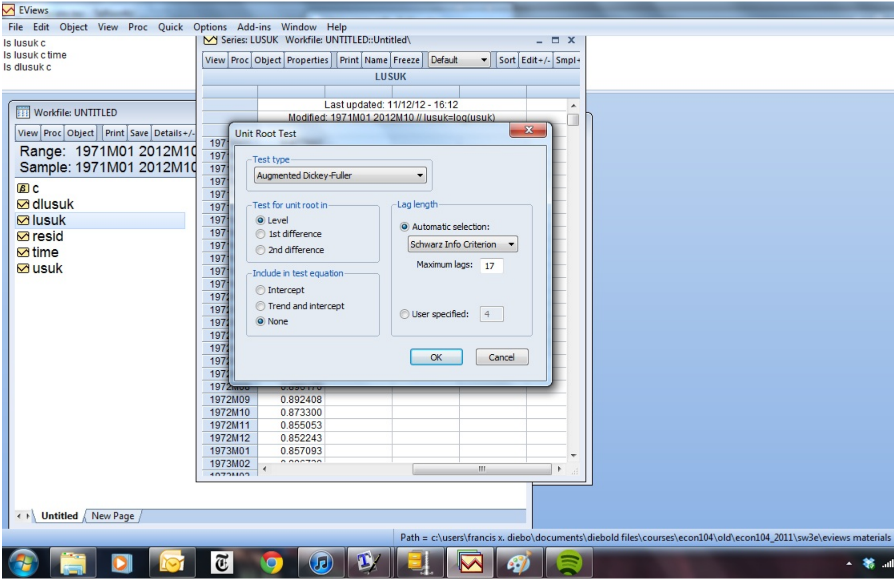
# Difference-Stationary Model (Random Walk With Drift)



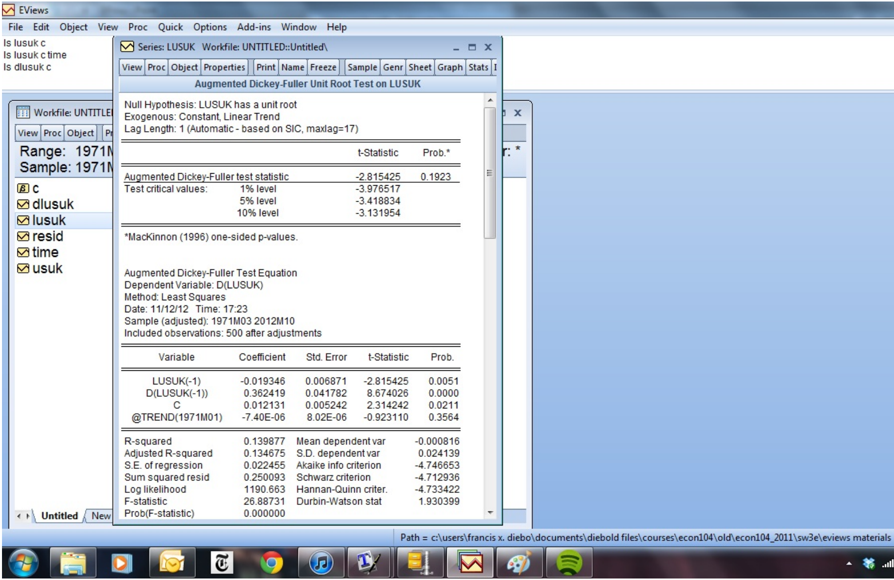
# Difference-Stationary Model (Random Walk With Drift)



# DF Tests – Option Screen



# ADF Test, Allowing for Trend Under the Alternative



*************************

# The Lag Operator

$$Ly_t = y_{t-1}$$

$AR(1)$ illustration:

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$y_t = \phi L y_t + \varepsilon_t$$

$$y_t - \phi L y_t = \varepsilon_t$$

$$(1 - \phi L) y_t = \varepsilon_t$$

$$\Phi(L) y_t = \varepsilon_t$$

$\Phi(L)$ is a polynomial of degree 1 in the L

# The Lag Operator, Continued ($AR(p)$)

$$y_t = \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t$$

$$y_t = \phi_1 L y_t + ... + \phi_p L^p y_t + \varepsilon_t$$

$$y_t - \phi_1 L y_t - ... - \phi_p L^p y_t = \varepsilon_t$$

$$(1 - \phi_1 L - ... - \phi_p L^p) y_t = \varepsilon_t$$

$$\Phi(L) y_t = \varepsilon_t$$

$\Phi(L)$ is a polynomial of degree $p$ in the lag operator

Roots of $\Phi(L)$ are important for nature and stability of dynamics

# Covariance Stationarity in $AR(p)$

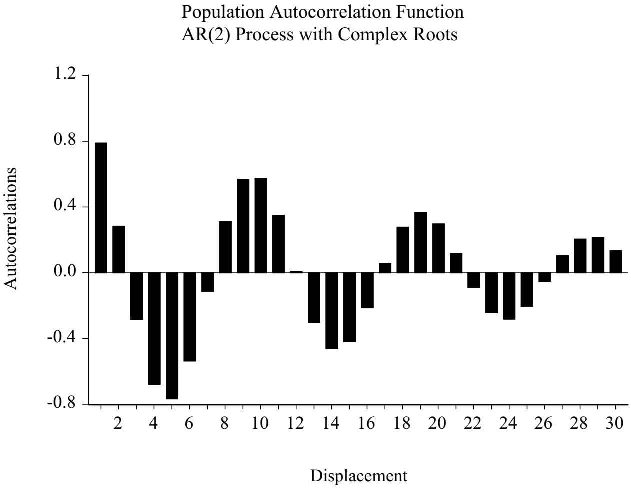
$AR(p)$ is

$$y_t = \phi_1 y_{t-1} + ... + \phi_p y_{t-p} + \varepsilon_t$$

$$(1 - \phi_1 L - ... - \phi_p L^p) y_t = \varepsilon_t$$

$$\Phi(L) y_t = \varepsilon_t$$

Stable if the $p$ roots of $\Phi(L)$ are outside the unit circle


Population Autocorrelation Function
AR(2) Process with Complex Roots
1.2
0.8
0.4
0.0
-0.4
-0.8
Autocorrelations
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
Displacement

# Big Data

# Selection, Shrinkage and Derived Inputs

"Data-rich" environments

"Wide data"

Dimensionality reduction is key: Selection, shrinkage, more.

# Selection Methods

- All Subsets

Quickly gets hard as there are $2^K$ subsets of $K$ regressors!

- Greedy Forward Selection

Start with intercept only and add the new regressor that minimizes RSS, then take the one variable model and add the new regressor that minimizes RSS, etc.

- Greedy Backward Selection

Start with $K$-variable model and remove the "least significant" variable, then take that $K-1$-variable model and remove the "least significant" variable, etc.

- Posterior odds and marginal likelihood:

$$\underbrace{\frac{p(M_i|y)}{p(M_j|y)}}_{posterior\ \ odds} = \underbrace{\frac{p(y|M_i)}{p(y|M_j)}}_{Bayes\ \ factor} \underbrace{\frac{p(M_i)}{p(M_j)}}_{prior\ \ odds}$$

- Information criteria:

# Model Selection by $MSE$ (or $R^2$)

$$MSE = \frac{\sum_{t=1}^{T} e_t^2}{T}$$

$$R^2 = 1 - \frac{\frac{1}{T}\sum_{t=1}^{T} e_t^2}{\frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})^2} = 1 - \frac{MSE}{\frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})^2}$$

Selection by $MSE$ (or $R^2$) produces in-sample over-fitting

# Model Selection by $s^2$ (or $\bar{R}^2$)

$$s^2 \;=\; \frac{1}{T-K}\sum_{t=1}^{T} e_t^2 \;=\; \left(\frac{T}{T-K}\right)\frac{\sum_{t=1}^{T} e_t^2}{T}$$

$$\bar{R}^2 \;=\; 1 \;-\; \frac{\frac{1}{T-K}\sum_{t=1}^{T} e_t^2}{\frac{1}{T-1}\sum_{t=1}^{T}(y_t-\bar{y})^2} \;=\; 1 \;-\; \frac{s^2}{\frac{1}{T-1}\sum_{t=1}^{T}(y_t-\bar{y})^2}$$

Selection by $s^2$ (or $\bar{R}^2$) *still* produces in-sample over-fitting

# Information Criteria for Model Selection

$$SIC \;=\; \left(T^{\left(\frac{k}{T}\right)}\right) \frac{\sum_{t=1}^{T} e_t^2}{T}$$

"Oracle property"
No over-fitting (asmptotically)!

$$AIC \;=\; \left(e^{\left(\frac{2k}{T}\right)}\right) \frac{\sum_{t=1}^{T} e_t^2}{T}$$

# Information Criteria for Model Selection

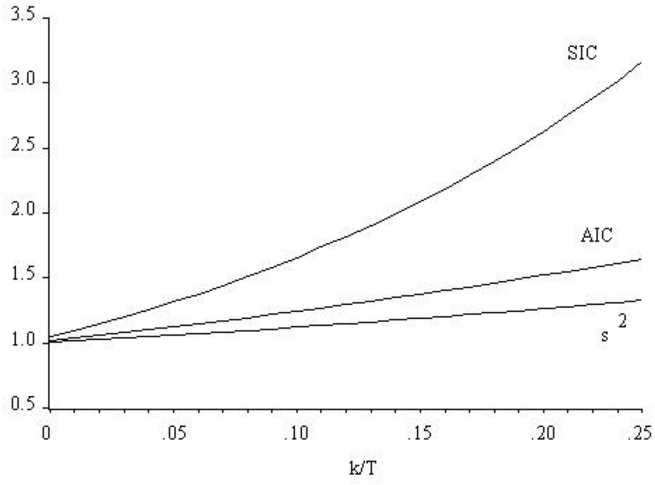



Figure: Degrees-of-Freedom Penalties

# Shrinkage Methods

- Bayesian regression:

$$\hat{\beta}_{bayes} = \omega_1 \hat{\beta}_{MLE} + \omega_2 \beta_0$$

- Ridge Regression:

$$\hat{\beta}_{ridge} = (X'X + \lambda I)^{-1} X'y$$

- Penalized regression:

$$\tilde{\beta} = argmin_{\beta_1 \ldots \beta_K} \left( \sum_{t=1}^{T} \left( y_t - \sum_{i=1}^{K} \beta_i x_{it} \right)^2 + \lambda \sum_{i=1}^{K} |\beta_i|^q \right)$$

– penalties smooth at the origin produce shrinkage
– penalties non-differentiable at the origin produce selection
– $q = 2$ is ridge; $q = 1$ is lasso; $q \to 0$ is selection.

# Aside: Review of Principal Components Analysis (PCA) for ata ($X$ Matrix) Description

Think of a wide $X$ matrix and how to "reduce" it.

$X'X$ eigendecomposition:

$$X'X = VD^2V'$$

The $j^{th}$ column of $V$, $v_j$, is the $j^{th}$ eigenvector of $X'X$
Diagonal matrix $D^2$ contains the descending eigenvalues of $X'X$

First principal component:

$$z_1 = Xv_1$$

$$var(z_1) = d_1^2/T$$

(maximal sample var among all possible l.c.'s of columns of $X$)

In general:

$$z_j = Xv_j \;\; \perp z_{j'}, j' \neq j$$

$$var(z_j) \leq d_j^2/T$$

# Derived Input Variable Methods I: PC Regression (PCR) and its First Problem

"Factor-Augmented Regression"
"Distill, then select then proceed"

Ridge and PCR are both shrinkage procedures.

BUT:

Ridge effectively includes all PC's and shrinks according to sizes of eigenvalues associated with the PC's.

PCR effectively shrinks some PCs completely to zero (those not included) and doesn't shrink others at all (those included).
– Awkward

# Derived Input Variable Methods I (Continued): PC Regression (PCR) and its Second Problem

No "supervision"

# Derived Input Variable Methods II: Partial Least Squares Regression (PLS)